LOCAL SELECTIVITY OF ORDERS IN CENTRAL SIMPLE ALGEBRAS

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Abstract. Let B be a central simple algebra of degree n over a number field K, and $L \subset B$ a strictly maximal subfield. We say that the ring of integers $\mathcal{O}_L$ is *selective* if there exists an isomorphism class of maximal orders in B no element of which contains $\mathcal{O}_L$. Many authors have worked to characterize the degree to which selectivity occurs, first in quaternion algebras, and more recently in higher-rank algebras. In the present work, we consider a local variant of the selectivity problem and applications.

We first prove a theorem characterizing which maximal orders in a local central simple algebra contain the global ring of integers $\mathcal{O}_L$ by leveraging the theory of affine buildings for $SL_r(D)$ where D is a local central division algebra. Then as an application, we use the local result and a local-global principle to show how to compute a set of representatives of the isomorphism classes of maximal orders in B, and distinguish those which are guaranteed to contain $\mathcal{O}_L$. Having such a set of representatives allows both algebraic and geometric applications. As an algebraic application, we recover a global selectivity result mentioned above, and give examples which clarify the interesting role of partial ramification in the algebra.

1. Introduction

Let B be a central simple algebra of degree n over a number field K, and $L \subset B$ a strictly maximal (i.e., $[L : K] = n$) subfield of B. There exists at least one maximal order $\mathcal{R}$ of B which contains the ring of integers $\mathcal{O}_L$, and so every element of the isomorphism class of $\mathcal{R}$ admits an embedding of $\mathcal{O}_L$. If there exists an isomorphism class of maximal orders in B no element of which contains $\mathcal{O}_L$, then $\mathcal{O}_L$ is said to be *selective*. This is equivalent to no element of the isomorphism class admitting an embedding of $\mathcal{O}_L$.

Many authors worked to characterize the degree to which selectivity occurs: [12], [10],[16], [22], [19] (in quaternion algebras), and [2], [20], [3], [4] (in higher-rank algebras). The tools which have been employed vary from the Bruhat-Tits tree in [12], to representation fields (a subfield of a spinor class field) in [3]. The results of [3] are very general, offering the proportion of isomorphism classes of maximal orders (an element of) which contain the order $\mathcal{O}_L$ (or any of its suborders), in terms of the index of the representation field in an associated spinor class field.

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In the present work, the authors continue their study (cf. [20]) of how the use of Bruhat-Tits buildings can illuminate problems for higher rank algebras as the Bruhat-Tits tree was used to answer selectivity questions in the quaternion case [12]. Locally, since all maximal orders are conjugate, every maximal order admits an embedding of $\mathcal{O}_L$, so a local selectivity question must be more discerning: to characterize the maximal orders in a local central simple algebra which contain $\mathcal{O}_L$. This turns out to be both an interesting and somewhat nuanced question.

To be more precise, we set some notation. By Wedderburn's structure theorem, we shall assume that $B = M_r(D)$ where D is a central division algebra over K of degree m, so that $n = rm$. Let ν be any place of K, and denote the completion of K at ν by K_ν, and when ν is a finite place, denote by $\mathcal{O}_\nu$ the valuation ring of K_ν. The completion of B at ν is given by

$$B_\nu = K_\nu \otimes_K B \cong M_{r_\nu}(D_\nu),$$

where D_ν is a central division algebra over K_ν of degree m_ν, so that $n = rm = r_\nu m_\nu$ with $r \mid r_\nu$. We say that a place ν of K *splits* in B if $m_\nu = 1$, *totally ramifies* in B if $m_\nu = n$, and *partially ramifies* in B if $1 < m_\nu < n$. If B totally ramifies at a finite place ν, there is a unique maximal order of B_ν, so of course $\mathcal{O}_L$ is contained in it, thus the only interest in local selectivity arises when ν is not totally ramified.

As a consequence of the condition that L is a strictly maximal subfield of B, we know that for each place ν of K and for all places $\mathfrak{P}$ of L lying above ν, $m_\nu \mid [L_\mathfrak{P} : K_\nu]$ (the Albert-Brauer-Hasse-Noether theorem), we have by (31.10) of [25], that each $L_\mathfrak{P}$ splits D_ν (and hence B_ν), and moreover by (28.5) of [25], for each place $\mathfrak{P}$ of L with $\mathfrak{P} \mid \nu$, there is a smallest integer $r_\mathfrak{P} \geq 1$ so that $L_\mathfrak{P}$ embeds in $M_{r_\mathfrak{P}}(D_\nu)$ as a K_ν-algebra; here $r_\mathfrak{P} = [L_\mathfrak{P} : K_\nu]/m_\nu$. Theorem 2.1 (which applies even in the quaternion case) says:

Theorem. *Let B be a central simple algebra over a number field K of dimension $n^2 \geq 4$ and L a degree n field extension of K which is contained in B. Let ν be a finite place of K which splits or is partially ramified in B, so $B_\nu = M_{r_\nu}(D_\nu)$ with $r_\nu > 1$, and D_ν a central division algebra over K_ν of degree m_ν. Assume that the place ν is unramified in L, and let $\{\mathfrak{P}_1, \ldots, \mathfrak{P}_g\}$ be the set of places of L lying above ν. As above, let $r_{\mathfrak{P}_i} = [L_{\mathfrak{P}_i} : K_\nu]/m_\nu$. Then $\mathcal{O}_L$ is contained in the maximal orders of B_ν represented by the homothety class $[\mathcal{L}] = [a_1, \ldots, a_{r_\nu}] \in \mathbb{Z}^{r_\nu}/\mathbb{Z}(1, \ldots, 1)$ if and only if there are $\ell_i \in \mathbb{Z}$ such that $[\mathcal{L}] = [\underbrace{\ell_1, \ldots, \ell_1}_{r_{\mathfrak{P}_1}}, \underbrace{\ell_2, \ldots, \ell_2}_{r_{\mathfrak{P}_2}}, \ldots, \underbrace{\ell_g, \ldots, \ell_g}_{r_{\mathfrak{P}_g}}]$.*

There are a number of applications of such a local result. As a primary application, it allows us to construct a set of representatives of all the isomorphism classes of maximal orders in the global algebra B, and specify those which are guaranteed to contain $\mathcal{O}_L$.

This is turn has at least two other applications, one algebraic and one geometric. In terms of the global selectivity problem, it allows one to compute not simply the selectivity proportion for $\mathcal{O}_L$, but distinguish those classes which necessarily admit an embedding of $\mathcal{O}_L$.

The same computation of explicit representatives of maximal orders in B also can be used in geometric realms such the development of a higher-dimensional analog of a construction of Vignerás [29] of isospectral non-isometric Riemann surfaces (e.g., [21]). Explicit characterization of these maximal orders allows the geometry of the corresponding manifold to be detailed, e.g., computation of the geodesic length spectrum.

As an algebraic application, we recover a global selectivity result mentioned above, and give an explicit example which demonstrates the effect of partial ramification in the algebra.

2. Local Results

Because our main application of these local results will be to construct a distinguished set of representatives for the isomorphism classes of maximal orders in the global algebra B, and then recover a selectivity result, we retain global notation throughout to allow for some dovetailing of local and global remarks.

Let ν be a finite place of K, and $B_\nu \cong M_{r_\nu}(D_\nu)$, with D_ν a central division algebra of degree m_ν over K_ν. Recall $[L : K] = n = \deg_K(B) = rm = r_\nu m_\nu$. We shall note in Theorem 4.3, if there is a finite place ν for which B_ν is a division algebra ($r_\nu = 1$), there can be no selectivity, so we assume for this section that $r_\nu > 1$.

Recall from the introduction that for each place ν of K, and each place $\mathfrak{P}$ of L lying above ν, $r_\mathfrak{P} = [L_\mathfrak{P} : K_\nu]/m_\nu \geq 1$ is the smallest integer so that $L_\mathfrak{P}$ embeds in $M_{r_\mathfrak{P}}(D_\nu)$ as a K_ν-algebra.

We note that

$$\sum_{\mathfrak{P}|\nu} r_\mathfrak{P} = \sum_{\mathfrak{P}|\nu} \frac{[L_\mathfrak{P} : K_\nu]}{m_\nu} = \frac{[L : K]}{m_\nu} = \frac{n}{m_\nu} = r_\nu,$$

and this means that

$$K_\nu \otimes_K L \cong \bigoplus_{\mathfrak{P}|\nu} L_\mathfrak{P} \hookrightarrow \bigoplus_{\mathfrak{P}|\nu} M_{r_\mathfrak{P}}(D_\nu) \hookrightarrow M_{r_\nu}(D_\nu), \tag{1}$$

with the last embedding as blocks along the diagonal.

We have fixed a global maximal order $\mathcal{R}$ in B which contains $\mathcal{O}_L$. We define completions $\mathcal{R}_\nu \subseteq B_\nu$ by:

$$\mathcal{R}_\nu = \begin{cases} \mathcal{O}_\nu \otimes_{\mathcal{O}_K} \mathcal{R} & \text{if } \nu \text{ is finite} \\ K_\nu \otimes_{\mathcal{O}_K} \mathcal{R} = B_\nu & \text{if } \nu \text{ is infinite.} \end{cases}$$

For finite places ν, we know by (17.3) of [25], that $\mathcal{R}_\nu$ is conjugate to $M_{r_\nu}(\Delta_\nu)$ where Δ_ν is the unique maximal order of D_ν, so we assume that B_ν has been identified with $M_{r_\nu}(D_\nu)$ in such a way that $\mathcal{R}_\nu = M_{r_\nu}(\Delta_\nu)$. Since all maximal orders of $M_{r_\mathfrak{P}}(D_\nu)$ are conjugate to $M_{r_\mathfrak{P}}(\Delta_\nu)$ we may, by a change of basis, adjust the embeddings $L_\mathfrak{P} \hookrightarrow M_{r_\mathfrak{P}}(D_\nu)$ so that the ring of integers $\mathcal{O}_\mathfrak{P} \subset M_{r_\mathfrak{P}}(\Delta_\nu)$. Now by Exercise 5.4 (p. 76) of [25], $\mathcal{O}_\nu$ is a faithfully flat

$\mathcal{O}_K$-module and the containment of $\mathcal{O}_L \subset \mathcal{R}$ extends to one of $\mathcal{O}_\nu \otimes_{\mathcal{O}_K} \mathcal{O}_L \subset \mathcal{O}_\nu \otimes_{\mathcal{O}_K} \mathcal{R} = \mathcal{R}_\nu$. We identify $\mathcal{O}_L$ with its image $1 \otimes \mathcal{O}_L$, so will simply write $\mathcal{O}_L \subset \mathcal{R}_\nu$. More precisely, we will identify $\mathcal{O}_L$ with its image in $\bigoplus_{\mathfrak{P}|\nu} \mathcal{O}_\mathfrak{P}$ via:

$$\mathcal{O}_L \subset \mathcal{O}_\nu \otimes_{\mathcal{O}_K} \mathcal{O}_L \hookrightarrow \bigoplus_{\mathfrak{P}|\nu} \mathcal{O}_\mathfrak{P} \subset \bigoplus_{\mathfrak{P}|\nu} M_{r_\mathfrak{P}}(\Delta_\nu) \subset M_{r_\nu}(\Delta_\nu) = \mathcal{R}_\nu, \tag{2}$$

where we are using the subset notation to identify the object and its image.

Fix a uniformizing parameter $\boldsymbol{\pi} = \boldsymbol{\pi}_{D_\nu}$ of the maximal order Δ_ν, and let $d_k^\ell = \mathrm{diag}(\underbrace{\boldsymbol{\pi}^\ell, \dots, \boldsymbol{\pi}^\ell}_{k}, 1, \dots, 1) \in M_{r_\nu}(D_\nu)$. Put

$$\mathcal{R}(k,\ell) := d_k^\ell \mathcal{R}_\nu d_k^{-\ell} = d_k^\ell M_{r_\nu}(\Delta_\nu) d_k^{-\ell} = \begin{pmatrix} M_k(\Delta_\nu) & \boldsymbol{\pi}^\ell M_{k\times r_\nu-k}(\Delta_\nu) \\ \boldsymbol{\pi}^{-\ell} M_{r_\nu-k\times k}(\Delta_\nu) & M_{r_\nu-k}(\Delta_\nu) \end{pmatrix} \subset M_{r_\nu}(D_\nu). \tag{3}$$

Note that $\mathcal{R}(0,\ell) = \mathcal{R}(r_\nu,\ell) = \mathcal{R}(k,0) = \mathcal{R}_\nu = M_{r_\nu}(\Delta_\nu)$. If we let $\mathfrak{P}_1, \dots, \mathfrak{P}_g$ denote all the places of L lying above ν, then from equations (2),(3) above, it is evident that for all $\ell_1, \dots, \ell_g \in \mathbb{Z}$,

$$\mathcal{O}_L \subset \mathcal{R}(r_{\mathfrak{P}_1}, \ell_1) \cap \mathcal{R}(r_{\mathfrak{P}_1} + r_{\mathfrak{P}_2}, \ell_2) \cap \cdots \cap \mathcal{R}(r_{\mathfrak{P}_1} + \cdots + r_{\mathfrak{P}_g}, \ell_g), \tag{4}$$

that is,

$$\mathcal{O}_L \subset \bigcap_{\ell_i \in \mathbb{Z}} \left[\mathcal{R}(r_{\mathfrak{P}_1}, \ell_1) \cap \mathcal{R}(r_{\mathfrak{P}_1} + r_{\mathfrak{P}_2}, \ell_2) \cap \cdots \cap \mathcal{R}(r_{\mathfrak{P}_1} + \cdots + r_{\mathfrak{P}_g}, \ell_g)\right] = \bigoplus_{\mathfrak{P}|\nu} M_{r_\mathfrak{P}}(\Delta_\nu) \subset \mathcal{R}_\nu. \tag{5}$$

2.1. **Affine buildings and type distance.** We now translate this to the language of affine buildings. By (17.4) of [25], we know that every maximal order in B_ν has the form $\mathrm{End}_{\Delta_\nu}(\Lambda)$ where Λ is a full (i.e., rank r_ν), free (left) Δ_ν-lattice in $D_\nu^{r_\nu}$. We recall that a maximal order is characterized completely by the homothety class of its associated lattice, and homothety classes of lattices in $D_\nu^{r_\nu}$ are a very concrete way in which to characterize the vertices of the affine building associated to $\mathrm{SL}_{r_\nu}(D_\nu)$ (see section 3 of [1], or [26] Ch.9, §2). We know $\mathrm{GL}_{r_\nu}(D_\nu)$ acts transitively on the free Δ_ν-lattices of rank r_ν and acts invariantly on the homothety classes. Using that the maximal order Δ_ν of D_ν is a discretely valued ring with $\boldsymbol{\pi} = \boldsymbol{\pi}_{D_\nu}$ a uniformizer, we put $\mathrm{ord}_{\boldsymbol{\pi}}$ to be the exponential valuation on D_ν. Then we note that $\mathrm{ord}_{\boldsymbol{\pi}}$ is trivial on the commutator $[D^\times, D^\times]$, so for each $g \in \mathrm{GL}_{r_\nu}(D_\nu)$, $\mathrm{ord}_{\boldsymbol{\pi}}(\det(g))$ is a well-defined integer, where $\det(\cdot)$ is the Dieudonné determinant. It is then natural to define the *type* of a vertex as an integer modulo r_ν as follows (see [26]). Let Λ be a (free of rank r_ν) Δ_ν-lattice whose homothety class is assigned the type 0. For another such lattice Γ, let g be any element of $\mathrm{GL}_{r_\nu}(D_\nu)$ so that $\Gamma = g(\Lambda)$. Then the class of Γ is assigned type $\mathrm{ord}_{\boldsymbol{\pi}}(\det(g)) \pmod{r_\nu}$, which is well-defined on the homothety class since we are viewing the type modulo r_ν.

The simplicial structure of the building associated to $\mathrm{SL}_{r_\nu}(D_\nu)$ is reflected through its vertex types. In particular, the r_ν vertices of any chamber have types 0 through $(r_\nu - 1)$. In relating the vertices, we utilize the invariant factor theory which applies to free Δ_ν-lattices of rank r_ν. Let Γ and Λ be two rank r_ν free Δ_ν-lattices. Since we are working with homothety classes, we may assume that $\Gamma \subseteq \Lambda$. By (17.7) of [25], given two such lattices, there exists a basis $\{e_1, \ldots, e_{r_\nu}\}$ of $D_\nu^{r_\nu}$ and rational integers $0 \leq a_1 \leq \cdots \leq a_r$ so that

$$\Lambda = \bigoplus_{i=1}^{r_\nu} \Delta_\nu e_i \quad \text{and} \quad \Gamma = \bigoplus_{i=1}^{r_\nu} \Delta_\nu \boldsymbol{\pi}^{a_i} e_i.$$

Suppose that $\mathcal{E} = \mathrm{End}_{\Delta_\nu}(\Lambda)$, and $\mathcal{E}' = \mathrm{End}_{\Delta_\nu}(\Gamma)$. Using the invariant factor decomposition above, we define the *type distance* $td_\nu(\mathcal{E}, \mathcal{E}')$ to be

$$td_\nu(\mathcal{E}, \mathcal{E}') = \sum_{i=1}^{r_\nu} a_i \pmod{r_\nu}.$$

We note that this definition depends only upon the homothety class of the lattices. While it is true that $td_\nu(\mathcal{E}, \mathcal{E}') \equiv -td_\nu(\mathcal{E}', \mathcal{E}) \pmod{r_\nu}$, our main concern will be when the type distance $td_\nu(\mathcal{E}, \mathcal{E}')$ is divisible by some integer, so the order will be of little consequence. This definition of type distance generalizes the one in [20], where whenever the algebra was not totally ramified, it was split, so that $r_\nu = n$ and $D_\nu = K_\nu$.

2.2. **Local selectivity.** Pick a basis $\{\alpha_1, \ldots, \alpha_{r_\nu}\}$ of $D_\nu^{r_\nu}$ (and hence in particular fix an apartment of the building associated to $SL_{r_\nu}(D_\nu)$), so that with respect to this basis, $\mathcal{R}_\nu = M_{r_\nu}(\Delta_\nu) = \mathrm{End}_{\Delta_\nu}(\Lambda)$, where $\Lambda = \oplus_{i=1}^{r_\nu} \Delta_\nu \alpha_i$. Also we have $\mathcal{R}(k, \ell) = \mathrm{End}_{\Delta_\nu}(\mathcal{M}(k, \ell))$ where $\mathcal{M}(k, \ell) = \oplus_{i=1}^{k} \boldsymbol{\pi}^\ell \Delta_\nu \alpha_i \oplus \oplus_{i=k+1}^{r_\nu} \Delta_\nu \alpha_i$ and $\boldsymbol{\pi}$ is our fixed uniformizer in Δ_ν. As usual, this maximal order in B_ν can be represented by the homothety class of the lattice $\mathcal{M}(k, \ell)$, $[\mathcal{M}(k, \ell)] := [\underbrace{\ell, \ldots, \ell}_{k}, 0, \ldots, 0] \in \mathbb{Z}^{r_\nu}/\mathbb{Z}(1, \ldots, 1)$. Observe that $\mathcal{R}(k, \ell)$ has type $k\ell \pmod{r_\nu}$.

With the notation fixed as above, we characterize precisely which maximal orders in this apartment contain $\mathcal{O}_L$. The theorem is valid even in the quaternion case ($n = 2$). We shall continue to assume $r_\nu > 1$ (i.e. ν not totally ramified in B), otherwise B_ν has a unique maximal order, which must clearly contain $\mathcal{O}_L$.

Theorem 2.1. *Let B be a central simple algebra over a number field K of dimension $n^2 \geq 4$ and L a degree n field extension of K which is contained in B. Let ν be a finite place of K which splits or is partially ramified in B, so $B_\nu = M_{r_\nu}(D_\nu)$ with $r_\nu > 1$, and D_ν a central division algebra over K_ν of degree m_ν. Assume that the place ν is unramified in L, and let $\{\mathfrak{P}_1, \ldots, \mathfrak{P}_g\}$ be the set of places of L lying above ν. As above, let $r_{\mathfrak{P}_i} = [L_{\mathfrak{P}_i} : K_\nu]/m_\nu$. Then $\mathcal{O}_L$ is contained in the maximal orders of B_ν represented by the* *homothety class* $[\mathcal{L}] = [a_1, \dots, a_{r_\nu}] \in \mathbb{Z}^{r_\nu}/\mathbb{Z}(1, \dots, 1)$ *if and only if there are* $\ell_i \in \mathbb{Z}$ *such that* $[\mathcal{L}] = [\underbrace{\ell_1, \dots, \ell_1}_{r_{\mathfrak{P}_1}}, \underbrace{\ell_2, \dots, \ell_2}_{r_{\mathfrak{P}_2}}, \dots, \underbrace{\ell_g, \dots, \ell_g}_{r_{\mathfrak{P}_g}}]$.

Proof of Theorem. Consider equation (5). We know that $\mathcal{O}_L$ is contained in $\mathcal{R}(r_{\mathfrak{P}_1}, \ell_1) \cap \mathcal{R}(r_{\mathfrak{P}_1} + r_{\mathfrak{P}_2}, \ell_2) \cap \dots \cap \mathcal{R}(r_{\mathfrak{P}_1} + \dots + r_{\mathfrak{P}_g}, \ell_g)$ for any choice of $\ell_i \in \mathbb{Z}$. These orders correspond to homothety classes of lattices $[\mathcal{M}(r_{\mathfrak{P}_1} + \dots + r_{\mathfrak{P}_i}, \ell_i)] = \ell_i[\mathcal{M}(r_{\mathfrak{P}_1} + \dots + r_{\mathfrak{P}_i}, 1)] = \ell_i[\underbrace{1, \dots, 1}_{r_{\mathfrak{P}_1} + \dots + r_{\mathfrak{P}_i}}, 0, \dots, 0]$ as an element of $\mathbb{Z}^{r_\nu}/\mathbb{Z}(1, \dots, 1)$. In [7], it is shown that walks in an apartment are consistent with the natural group action on $\mathbb{Z}^{r_\nu}/\mathbb{Z}(1, \dots, 1)$, and by [28] the intersection of any finite number of maximal orders (containing $\Delta_\nu^{r_\nu}$) in an apartment is the same as the intersection of all the maximal orders in the convex hull they determine. The references above discuss the case where $D_\nu = K_\nu$, but the arguments generalize trivially to the setting of a vector space over D_ν instead of K_ν, as does the theory of buildings. Using these observations, we deduce that $\mathcal{O}_L$ is contained in maximal orders corresponding to

$$\begin{aligned}[]&[\mathcal{M}(r_{\mathfrak{P}_1}, \ell_1) + \mathcal{M}(r_{\mathfrak{P}_1} + r_{\mathfrak{P}_2}, \ell_2) + \dots + \mathcal{M}(r_{\mathfrak{P}_1} + \dots + r_{\mathfrak{P}_g}, \ell_g)] = \\ &\qquad [\underbrace{\ell_1 + \dots + \ell_g, \dots, \ell_1 + \dots + \ell_g}_{r_{\mathfrak{P}_1}}, \underbrace{\ell_2 + \dots + \ell_g, \dots, \ell_2 + \dots + \ell_g}_{r_{\mathfrak{P}_2}}, \dots, \underbrace{\ell_g, \dots, \ell_g}_{r_{\mathfrak{P}_g}}].\end{aligned}$$

Since the $\ell_i \in \mathbb{Z}$ are arbitrary, a simple change of variable ($\ell_k + \dots + \ell_g \mapsto \ell_k$) shows that $\mathcal{O}_L$ is contained in the maximal orders specified in the proposition. We now show these are the only maximal orders in the apartment which contain $\mathcal{O}_L$. To proceed, we need to set some notation and prove a technical lemma.

For any place $\mathfrak{P}$ of L lying above ν, we have (by assumption) that $L_{\mathfrak{P}}/K_\nu$ is an unramified extension of degree $f := r_{\mathfrak{P}} m_\nu$. Let $\overline{\mathcal{O}}_{\mathfrak{P}}$ and $\overline{\mathcal{O}}_\nu$ be the associated residue fields and let $q = |\overline{\mathcal{O}}_\nu|$. Now let ω be a primitive $q^f - 1$ root of unity over K_ν, so that $L_{\mathfrak{P}} = K_\nu(\omega)$. We know that D_ν contains an inertia field, W_ν, which is unique up to conjugacy. It is an unramified extension of K_ν and a maximal subfield of D_ν, having degree $[W_\nu : K_\nu] = m_\nu$. Without loss, we may assume that $K_\nu \subseteq W_\nu \subseteq L_{\mathfrak{P}}$, with W_ν generated over K_ν by an appropriate power of ω (since $q^{m_\nu - 1} \mid q^f - 1$). Now let $h = \min_{K_\nu}(\omega)$ be the minimal polynomial of ω over K_ν. As ω is integral, we know $h \in \mathcal{O}_\nu[x]$.

Proposition 2.2. *From Theorem 5.10 and Corollary 5.11 of* [25]*, we recall*

- $\mathcal{O}_{\mathfrak{P}} = \mathcal{O}_\nu[\omega]$; $\quad \overline{\mathcal{O}}_{\mathfrak{P}} = \overline{\mathcal{O}}_\nu[\overline{\omega}]$.
- $\overline{h} = \min_{\overline{\mathcal{O}}_\nu}(\overline{\omega})$ *and is separable.*
- $L_{\mathfrak{P}}/K_\nu$ *and* $\overline{\mathcal{O}}_{\mathfrak{P}}/\overline{\mathcal{O}}_\nu$ *are cyclic extensions with isomorphic Galois groups.*

The technical lemma we require is:

Lemma 2.3. *Let R be the valuation ring of ν in K, and S its integral closure in L. Suppose that $\mathcal{E}$ is a ring containing both R and $\mathcal{O}_L$. Then $S \subset \mathcal{E}$.*

Proof. By Corollary 5.22 of [6], S is the intersection of all valuation rings of L which contain R. The valuation ring R is equal to the localization $D^{-1}\mathcal{O}_K$ where $D = \mathcal{O}_K \setminus \nu\mathcal{O}_K$, and the valuation rings of L which contain R are precisely the localizations of $\mathcal{O}_L$ at the places $\mathfrak{P}_1, \ldots, \mathfrak{P}_g$ of L which lie above ν. By p43 of [25], the intersection of these localizations, S, is equal to the localization $T^{-1}\mathcal{O}_L$, where $T = \mathcal{O}_L \setminus (\mathfrak{P}_1 \cup \cdots \cup \mathfrak{P}_g)$.

It is easy to see that $D \subseteq T$ since if $\alpha \in D = \mathcal{O}_K \setminus \nu\mathcal{O}_K$, we have $\alpha \in \mathcal{O}_L$ and if $\alpha \in \mathfrak{P}_i$ for some i, then $\alpha \in \mathfrak{P}_i \cap \mathcal{O}_K = \nu\mathcal{O}_K$, a contradiction. So $D^{-1}\mathcal{O}_L \subseteq T^{-1}\mathcal{O}_L$. To show equality, we need only show that for $\beta \in T$, $\beta^{-1} \in D^{-1}\mathcal{O}_L$. Since $\beta \in \mathcal{O}_L$ we know that $N_{L/K}(\beta) := \beta\tilde{\beta} \in \mathcal{O}_K$, which means $\tilde{\beta} = \beta^{-1}N_{L/K}(\beta) \in L$ and is integral, hence in $\mathcal{O}_L$, so $\beta^{-1} = \tilde{\beta}/N_{L/K}(\beta)$, and we need only check that $N_{L/K}(\beta) \in D$. Suppose to the contrary that $N_{L/K}(\beta) \in \nu\mathcal{O}_K$. Then we show that $\nu \in \mathfrak{P}_i$ for some i, a contradiction. We use the extension of the norm to ideals and that $N_{L/K}(\beta\mathcal{O}_L) = N_{L/K}(\beta)\mathcal{O}_K$. If we write $\beta\mathcal{O}_L = \mathfrak{P}_1^{m_1} \cdots \mathfrak{P}_g^{m_g}\mathfrak{Q}$ where $\mathfrak{Q}$ is an ideal place to the $\mathfrak{P}_i$, and let $\mathfrak{q} = \mathfrak{Q} \cap \mathcal{O}_K$, then $N_{L/K}(\beta\mathcal{O}_L) = (\nu\mathcal{O}_K)^{\sum_{i=1}^{g} m_i f_i}\mathfrak{q}^f$ where $f_i = f(\mathfrak{P}_i : \nu)$ and $f = f(\mathfrak{Q} : \mathfrak{q})$ are the corresponding inertial degrees. So $N_{L/K}(\beta) \in \nu\mathcal{O}_K$ if and only if some $m_i > 0$ which is to say that $\beta \in \mathfrak{P}_i$, a contradiction. Thus we have that S, the integral closure of R in L, can be expressed as $D^{-1}\mathcal{O}_L = R \cdot \mathcal{O}_L$, so any ring $\mathcal{E}$ containing R and $\mathcal{O}_L$ contains S. $\square$

Continuing now with the proof of Theorem 2.1, denote $\mathfrak{p}$ denote the two-sided ideal $\boldsymbol{\pi}\Delta_\nu$ of Δ_ν, and suppose that $\mathcal{O}_L$ is contained in a maximal order $\Lambda(a_1, \ldots, a_{r_\nu})$ where

$$\Lambda(a_1, \ldots, a_{r_\nu}) = \operatorname{diag}(\boldsymbol{\pi}^{a_1}, \ldots, \boldsymbol{\pi}^{a_{r_\nu}})M_{r_\nu}(\Delta_\nu)\operatorname{diag}(\boldsymbol{\pi}^{a_1}, \ldots, \boldsymbol{\pi}^{a_{r_\nu}})^{-1} =$$
$$\begin{pmatrix} \Delta_\nu & \mathfrak{p}^{a_1-a_2} & \mathfrak{p}^{a_1-a_3} & \cdots & \mathfrak{p}^{a_1-a_{r_\nu}} \\ \mathfrak{p}^{a_2-a_1} & \Delta_\nu & \mathfrak{p}^{a_2-a_3} & \cdots & \mathfrak{p}^{a_2-a_{r_\nu}} \\ \mathfrak{p}^{a_3-a_1} & \mathfrak{p}^{a_3-a_2} & \ddots & \cdots & \mathfrak{p}^{a_3-a_{r_\nu}} \\ \vdots & \vdots & & \Delta_\nu & \vdots \\ \mathfrak{p}^{a_{r_\nu}-a_1} & \cdots & & \mathfrak{p}^{a_{r_\nu}-a_{r_\nu-1}} & \Delta_\nu \end{pmatrix},$$

that is $\Lambda(a_1, \ldots, a_{r_\nu})$ corresponds to the homothety class of the lattice $[a_1, \ldots, a_{r_\nu}]$ relative to our fixed basis $\{\alpha_1, \ldots, \alpha_{r_\nu}\}$ of $D_\nu^{r_\nu}$. By equation (5), we can reorder subsets of the basis $\{\alpha_1, \ldots, \alpha_{r_{\mathfrak{P}_1}}\}$, $\{\alpha_{r_{\mathfrak{P}_1}+1}, \ldots, \alpha_{r_{\mathfrak{P}_1}+r_{\mathfrak{P}_2}}\}$, $\ldots$, $\{\alpha_{r_{\mathfrak{P}_1}+\cdots+r_{\mathfrak{P}_{g-1}}+1}, \ldots, \alpha_{r_\nu}\}$ so that equation (5) remains valid and $a_1 \leq \cdots \leq a_{r_{\mathfrak{P}_1}}$, $a_{r_{\mathfrak{P}_1}+1} \leq \cdots \leq a_{r_{\mathfrak{P}_1}+r_{\mathfrak{P}_2}}$, $\ldots$, $a_{r_{\mathfrak{P}_1}+\cdots+r_{\mathfrak{P}_{g-1}}+1} \leq \cdots \leq a_{r_\nu}$.

Now we assume that $[a_1, \ldots, a_{r_\nu}]$ is not of the form $[\underbrace{\ell_1, \ldots, \ell_1}_{r_{\mathfrak{P}_1}}, \underbrace{\ell_2, \ldots, \ell_2}_{r_{\mathfrak{P}_2}}, \ldots, \underbrace{\ell_g, \ldots, \ell_g}_{r_{\mathfrak{P}_g}}]$ for $\ell_i \in \mathbb{Z}$. Since we can permute the order in which we list the places $\mathfrak{P}_i$ of L lying above ν, we may assume that there is an r_0 with $1 \leq r_0 < r_{\mathfrak{P}_1}$ so that $a_1 = \cdots = a_{r_0} < a_{r_0+1} \leq \cdots \leq a_{r_{\mathfrak{P}_1}}$.

From equation (2), we know that $\mathcal{O}_L \subset \oplus_{i=1}^g \mathcal{O}_{\mathfrak{P}_i} \subset \oplus_{i=1}^g M_{r_{\mathfrak{P}_i}}(\Delta_\nu) \subset M_{r_\nu}(\Delta_\nu)$, so

$$\mathcal{O}_L \subset \oplus_{i=1}^g M_{r_{\mathfrak{P}_i}}(\Delta_\nu) \cap \Lambda(a_1, \dots, a_{r_\nu}) =: \Gamma = \left(\begin{array}{c|c|c} \Lambda_1 & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & \Lambda_g \end{array}\right),$$

where the $\Lambda_i \subset M_{r_{\mathfrak{P}_i}}(\Delta_\nu)$. Since $\mathcal{O}_\nu$ (as scalar matrices) and $\mathcal{O}_L$ are contained in Γ, Lemma 2.3 gives us that S, the integral closure of $R = (\mathcal{O}_\nu \cap K)$ in L, is contained in Γ. Thus $\mathcal{O}_\nu \otimes_R S \subset \mathcal{O}_\nu \otimes_R \Gamma = \Gamma$. By Proposition II.4 of [27], $\mathcal{O}_\nu \otimes_R S \cong \oplus_{i=1}^g \mathcal{O}_{\mathfrak{P}_i}$, so from $\mathcal{O}_\nu \otimes_R S \subset \Gamma$, we may assume that $\mathcal{O}_{\mathfrak{P}_1} \hookrightarrow \Lambda_1$, from which we shall derive a contradiction.

So we focus on Λ_1, the upper $r_{\mathfrak{P}_1} \times r_{\mathfrak{P}_1}$ block of $\oplus_{i=1}^g M_{r_{\mathfrak{P}_i}}(\Delta_\nu) \cap \Lambda(a_1, \dots, a_{r_\nu})$. That intersection is contained in

$$\Gamma_1 := \left(\begin{array}{c|c} M_{r_0}(\Delta_\nu) & M_{r_0 \times r_{\mathfrak{P}_1} - r_0}(\Delta_\nu) \\ \hline \boldsymbol{\pi} M_{r_{\mathfrak{P}_1} - r_0 \times r_0}(\Delta_\nu) & M_{r_{\mathfrak{P}_1} - r_0}(\Delta_\nu) \end{array}\right). \tag{6}$$

Write $\mathfrak{P}$ for $\mathfrak{P}_1$. As in Proposition 2.2 and the discussion which immediately precedes it, we write $L_{\mathfrak{P}} = K_\nu(\omega)$ ($\mathcal{O}_{\mathfrak{P}} = \mathcal{O}_\nu[\omega]$) where ω is an appropriate primitive root of unity over K_ν, and h is its minimal polynomial over K_ν. We know that $h \in \mathcal{O}_\nu[x]$ is monic and irreducible of degree $[L_{\mathfrak{P}} : K_\nu] = r_{\mathfrak{P}} m_\nu$. Under the embedding $\mathcal{O}_{\mathfrak{P}} \hookrightarrow \Gamma_1$ we send $\omega \mapsto \gamma \in \Gamma_1$. In particular, $h(\gamma) = 0$.

Case 1: $m_\nu = 1$ (ν splits in B), which means $D_\nu = K_\nu$, $\Delta_\nu = \mathcal{O}_\nu$, and $\mathfrak{p} = \pi\mathcal{O}_\nu$. Let $\chi_\gamma = \det(xI - \gamma)$ denote the characteristic polynomial of $\gamma \in \Gamma_1 \subset M_{r_{\mathfrak{P}}}(\mathcal{O}_\nu)$. Since $\deg(\chi_\gamma) = r_{\mathfrak{P}} = \deg(h)$ and $\chi_\gamma(\gamma) = 0$, and h is irreducible, we have $h \mid \chi_\gamma$, hence $h = \chi_\gamma$ by comparing degrees. On the other hand viewing $\chi_\gamma \pmod{\pi\mathcal{O}_\nu}$ means computing the characteristic polynomial in $\Gamma_1 \pmod{\pi\mathcal{O}_\nu} \subset M_{r_{\mathfrak{P}}}(\overline{\mathcal{O}}_\nu)$, whose block structure will make χ_γ reducible mod $\pi\mathcal{O}_\nu$. If $\overline{h} = \overline{\chi}_\gamma = \overline{h}_1\overline{h}_2$ with $\gcd(\overline{h}_1, \overline{h}_2) = 1$, then we get a nontrivial factorization of h over $\mathcal{O}_\nu$ by Hensel's lemma, a contradiction to the irreducibility of h. If not, then $\overline{h} = (\overline{h}_0)^k$ for some irreducible $h_0 \in \overline{\mathcal{O}}_\nu[x]$ with $\deg(\overline{h}_0) < \deg(\overline{h})$. But this means that $\overline{h}$ has multiple roots, contrary to Proposition 2.2.

Case 2: $m_\nu > 1$. Now $\deg(h) = r_{\mathfrak{P}} m_\nu$, and $\gamma \in \Gamma_1 \subset M_{r_{\mathfrak{P}}}(\Delta_\nu)$. As above, let W_ν be a maximal unramified extension of K_ν contained in $L_{\mathfrak{P}} \cap D_\nu$; recall $[W_\nu : K_\nu] = m_\nu$. As a maximal subfield of D_ν, W_ν is a splitting field for D_ν and we consider $1 \otimes \gamma \in M_{r_{\mathfrak{P}} \cdot m_\nu}(W_\nu)$. By Theorem 9.3 of [25] the characteristic polynomial $\chi_{1\otimes\gamma} \in \mathcal{O}_\nu[x]$, which is to say it is independent of the splitting field for D_ν. As in the previous case, we deduce that $h = \chi_{1\otimes\gamma}$. To maintain the flow of this argument, we defer the proof of the following lemma to the end of this proof.

Lemma 2.4. *$\overline{\chi}_{1\otimes\gamma}$ is reducible in $\overline{\mathcal{O}}_{W_\nu}[x]$. In particular, $\overline{\chi}_{1\otimes\gamma} = \overline{h}_1\overline{h}_2$ with $\overline{h}_i \in \overline{\mathcal{O}}_{W_\nu}[x]$ and $\deg(\overline{h}_1) = r_0 < r_{\mathfrak{P}}$.*

If $\overline{h} = \overline{\chi}_{1\otimes\gamma} = (\overline{h}_0)^k$ with $\deg(\overline{h}_0) < \deg(\overline{h})$, then as in the previous case $\overline{h}$ has multiple roots, a contradiction. On the other hand, if $\overline{h}$ factors into relatively prime factors, Hensel's lemma will only provide a nontrivial factorization over $\mathcal{O}_{W_\nu}$ which is actually expected since h is irreducible over K_ν and $[W_\nu : K_\nu] = m_\nu > 1$. So we need to dig a bit deeper. Let $G = \mathrm{Gal}(L_{\mathfrak{P}}/K_\nu)$ and $H = \mathrm{Gal}(L_{\mathfrak{P}}/W_\nu)$. Then

$$h = \min_{K_\nu}(\omega) = \prod_{\sigma\in G}(x - \sigma(\omega)) = \prod_{\sigma\in G/H}\prod_{\tau\in H}(x - \tau\sigma(w)).$$

Let $h_\sigma = \prod_{\tau\in H}(x - \tau\sigma(w))$. Since $h_\sigma^\tau = h_\sigma$ for all $\tau \in H$, by Galois theory we have that $h_\sigma \in \mathcal{O}_{W_\nu}[x]$, and $\deg(h_\sigma) = |H| = [L_{\mathfrak{P}} : W_\nu] = r_{\mathfrak{P}}$. Moreover since $L_{\mathfrak{P}} = K_\nu(\omega) = K_\nu(\sigma(w))$ for any $\sigma \in G$, $[L_{\mathfrak{P}} : W_\nu] = \deg(\min_{W_\nu}(\sigma(\omega))$, we see that $h_\sigma = \min_{W_\nu}(\sigma(\omega))$, and so in particular, $h = \prod_{\sigma\in G/H} h_\sigma$ is the irreducible factorization of h in $\mathcal{O}_{W_\nu}[x]$.

Now consider $\overline{h} \in \overline{\mathcal{O}}_\nu[x] \subset \overline{\mathcal{O}}_{W_\nu}[x]$. We have that $\overline{h} = \prod_{\sigma\in G/H}\overline{h}_\sigma$ and $\overline{h}_\sigma \in \overline{\mathcal{O}}_{W_\nu}[x]$. Recall that $\overline{h} = \min_{\overline{\mathcal{O}}_\nu}(\overline{\omega})$ and the isomorphisms $G = \mathrm{Gal}(L_{\mathfrak{P}}/K_\nu) \cong \mathrm{Gal}(\overline{\mathcal{O}}_{\mathfrak{P}}/\overline{\mathcal{O}}_\nu)$ and $H = \mathrm{Gal}(L_{\mathfrak{P}}/W_\nu) \cong \mathrm{Gal}(\overline{\mathcal{O}}_{\mathfrak{P}}/\overline{\mathcal{O}}_{W_\nu})$ give that the decomposition $\overline{h} = \prod_{\sigma\in G/H}\overline{h}_\sigma$ is the irreducible factorization of $\overline{h}$ in $\overline{\mathcal{O}}_{W_\nu}[x]$. But this contradicts Lemma 2.4 which says that $\overline{h} = \overline{\chi}_{1\otimes\gamma}$ has a factor of degree $s < r_{\mathfrak{P}}$. □

Proof of Lemma 2.4. To set the notation, we have $\Gamma_1 \subset M_{r_{\mathfrak{P}}}(\Delta_\nu)$. Following §14 of [25], we can choose $\boldsymbol{\pi} \in \Delta_\nu$ a uniformizer with $\boldsymbol{\pi}^{m_\nu} = \pi_\nu$ (π_ν a uniformizer in K_ν), and let ω_0 be a primitive $q^{m_\nu} - 1$ root of unity over K_ν, $q = |\overline{\mathcal{O}}_\nu|$. So $W_\nu = K_\nu(\omega_0)$ is an unramified extension of K_ν in D_ν with degree m_ν over K_ν. Then

$$\Delta_\nu = \bigoplus_{i,j=0}^{m_\nu - 1}\mathcal{O}_\nu\omega_0^i\boldsymbol{\pi}^j = \mathcal{O}_\nu[\omega_0, \boldsymbol{\pi}]; \qquad D_\nu = K_\nu[\omega_0, \boldsymbol{\pi}].$$

In (14.6) [25], Reiner gives an explicit K_ν-isomorphism

$$D_\nu \to M_{m_\nu}(W_\nu) \cong W_\nu \otimes_{K_\nu} D_\nu \text{ denoted simply } a \mapsto a^*.$$

From (14.7) [25], we see that for $a \in \Delta_\nu$, $a^* \in M_{m_\nu}(\mathcal{O}_{W_\nu})$ has upper triangular image in $M_{m_\nu}(\overline{\mathcal{O}}_{W_\nu})$, and for $a \in \boldsymbol{\pi}\Delta_\nu$, a^* has strictly upper triangular image in $M_{m_\nu}(\overline{\mathcal{O}}_{W_\nu})$. The map $a \mapsto a^*$ now extends linearly to $M_{r_{\mathfrak{P}}}(D_\nu) \to M_{r_{\mathfrak{P}}\cdot m_\nu}(W_\nu)$.

We first work through a simple, but non-trivial example which will make the general proof much easier to understand.

Example 2.5. Let $r_0 = 3$, $m_\nu = 2$, and $r_{\mathfrak{P}} > r_0$ (the exact value will not matter). Then

$$\gamma \in \Gamma_1 = \left(\begin{array}{c|c} M_3(\Delta_\nu) & M_{3\times r_{\mathfrak{p}}-3}(\Delta_\nu) \\ \hline \boldsymbol{\pi}M_{r_{\mathfrak{P}}-3\times 3}(\Delta_\nu) & M_{r_{\mathfrak{p}}-3}(\Delta_\nu) \end{array}\right).$$

Then $\overline{\chi}_{1\otimes\gamma} = \det(-A)$ (the minus is for easier typesetting), where

$$
A = \left[\begin{array}{cc|cc|cc||cc|cc}
a_{11}-x & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & \dots & \dots \\
0 & a_{22}-x & 0 & a_{24} & 0 & a_{26} & 0 & a_{28} & \dots & \dots \\
\hline
a_{31} & a_{32} & a_{33}-x & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & \dots & \dots \\
0 & a_{42} & 0 & a_{44}-x & 0 & a_{46} & 0 & a_{48} & \dots & \dots \\
\hline
a_{51} & a_{52} & a_{53} & a_{54} & a_{55}-x & a_{56} & a_{57} & a_{58} & \dots & \dots \\
0 & a_{62} & 0 & a_{64} & 0 & a_{66}-x & 0 & a_{68} & \dots & \dots \\
\hline\hline
0 & * & 0 & * & 0 & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
\hline
0 & * & 0 & * & 0 & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
0 & * & 0 & * & 0 & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & *
\end{array}\right]. \tag{7}
$$

We are going to compute this determinant using minors with expansions focusing on columns 1, 3, 5 where the entries in the lower left blocks are all zero. By expanding, we find that after three iterations, all of the summands in the determinant will be contain the determinant of the same $(r_\mathfrak{p}-3)\times(r_\mathfrak{p}-3)$ minor. Collecting the other terms gives a degree $r_0 = 3$ factor. The notation we shall use is $A(i_1,\dots,i_r|j_1,\dots j_s)$ will denote the matrix obtained from A be removing rows $i_1, i_2, \dots, i_r$ and columns $j_1, j_2, \dots, j_s$.

Expanding along the first column, we obtain:

$$\det(A) = (a_{11}-x)\det A(1|1) + a_{31}\det A(3|1) + a_{51}\det A(5|1).$$

In computing $\det A(m|1)$, we now look at what would be column 3 of the original matrix A which now has only two non-zero entries in that column of the minor.

$$
\begin{aligned}
\det A(1|1) &= (a_{33}-x)\det A(1,3|1,3) + a_{53}\det A(1,5|1,3),\\
\det A(3|1) &= a_{13}\det A(1,3|1,3) - a_{53}\det A(3,5|1,3),\\
\det A(5|1) &= -a_{13}\det A(1,3|1,3) - (a_{33}-x)\det A(3,5|1,3).
\end{aligned}
$$

In this last stage we need to compute the determinant of three minors, and the expression for each will be a multiple of $\det A(1,3,5|1,3,5)$ from which we will obtain the claim.

$$
\begin{aligned}
\det A(1,3|1,3) &= (a_{55}-x)\det A(1,3,5|1,3,5),\\
\det A(1,5|1,3) &= -a_{35}\det A(1,3,5|1,3,5),\\
\det A(3,5|1,3) &= a_{15}\det A(1,3,5|1,3,5).
\end{aligned}
$$

Now by inspection we see that we obtain a product of a cubic and a factor of degree $r_{\mathfrak{P}} - 3$. $\square$

We now turn to the general case. We have a

$$\gamma \in \Gamma_1 := \left(\begin{array}{c|c} M_{r_0}(\Delta_\nu) & M_{r_0 \times r_{\mathfrak{P}_1} - r_0}(\Delta_\nu) \\ \hline \boldsymbol{\pi} M_{r_{\mathfrak{P}_1} - r_0 \times r_0}(\Delta_\nu) & M_{r_{\mathfrak{P}_1} - r_0}(\Delta_\nu) \end{array} \right).$$

Then $1 \otimes \gamma \in W_\nu \otimes_{K_\nu} \Gamma_1 \subset M_{r_{\mathfrak{P}} m_\nu}(\mathcal{O}_{W_\nu})$, with reduced characteristic polynomial $\chi_{1\otimes\gamma} \in \mathcal{O}_{W_\nu}[x]$ of degree $r_{\mathfrak{P}} m_\nu$. Then the reduction, $\overline{\chi}_{1\otimes\gamma}$, of the characteristic polynomial modulo $\pi_\nu \mathcal{O}_{W_\nu}$ is given as in the example above as $\overline{\chi}_{1\otimes\gamma} = \det(-A)$, where A has entries in $\overline{\mathcal{O}}_{W_\nu}[x]$ and is given by (using s for $r_0 - 1$, m for m_ν and writing $a_{i,j}$ instead of a_{ij} for clarity)

$$\left[\begin{array}{ccc|ccc|c|ccc||cc}
a_{1,1}-x & \cdots & a_{1,m} & a_{1,m+1} & \cdots & a_{1,2m} & \cdots & a_{1,sm+1} & \cdots & a_{1,r_0m} & a_{1,r_0m+1} & \cdots \\
0 & \ddots & \vdots & 0 & \ddots & \vdots & \cdots & 0 & \ddots & \vdots & \vdots & \\
0 & \cdots & a_{m,m}-x & 0 & \cdots & a_{m,2m} & \cdots & 0 & \cdots & a_{m,r_0m} & \vdots & \\
\hline
a_{m+1,1} & \cdots & a_{m+1,m} & a_{m+1,m+1}-x & \cdots & a_{m+1,2m} & \cdots & a_{m+1,sm+1} & \cdots & a_{m+1,r_0m} & \vdots & \\
0 & \ddots & \vdots & 0 & \ddots & \vdots & \cdots & 0 & \ddots & \vdots & \vdots & \\
0 & \cdots & a_{2m,m} & 0 & \cdots & a_{2m,2m}-x & \cdots & 0 & \cdots & a_{2m,r_0m} & \vdots & \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \\
\hline
a_{sm+1,1} & \cdots & a_{sm+1,m} & a_{sm+1,m+1} & \cdots & a_{sm+1,2m} & \cdots & a_{sm+1,sm+1}-x & \cdots & a_{sm+1,r_0m} & \vdots & \\
0 & \ddots & \vdots & 0 & \ddots & \vdots & \cdots & 0 & \ddots & \vdots & \vdots & \\
0 & \cdots & a_{r_0m,m} & 0 & \cdots & a_{r_0m,2m} & \cdots & 0 & \cdots & a_{r_0m,r_0m}-x & \vdots & \\
0 & * & * & 0 & * & * & \cdots & 0 & * & * & a_{r_0m+1,r_0m+1}-x & \\
\vdots & \ddots & * & \vdots & \ddots & * & \cdots & 0 & \ddots & * & 0 & \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & \ddots \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \\
\hline
0 & * & * & 0 & * & * & \cdots & 0 & * & * & * & \\
\vdots & \ddots & * & \vdots & \ddots & * & \cdots & 0 & \ddots & * & 0 & \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & \ddots
\end{array} \right]. \tag{8}$$

We are going to partially compute this determinant, taking advantage of the zeros in columns $km_\nu + 1$, $k = 0, \ldots, (r_0 - 1)$ (below row $r_0 m_\mu$). The goal is to indicate that after r_0 iterations, every minor will have the same form, and the determinant of this minor will be therefore be a factor of the reduced characteristic polynomial (viewed over the residue field).

Computing the determinant by expanding along the first column, we obtain (still using $s = r_0 - 1$, m for m_ν, and writing $a_{i,j}$ for a_{ij} for clarity):

$$\det(A) = (a_{1,1} - x)\det A(1|1) + \sum_{k=1}^{s} a_{km+1,1} \det A(km+1|1)$$

So at this stage our determinant involves the determinants of new minors of the form $\det A(km+1|1)$, $k = 0, \ldots, s$, that is over column 1 and all the rows with nontrivial entries.

In computing each term $\det A(*|1)$, we next want to expand along what would be column $m+1$ of the original matrix A which now has only $r_0 - 1$ non-zero entries in that column of the minor. The final simplification we make is that we shall not fuss about the correct signs of each summand in the expression of the determinant since they will be immaterial in the end, so we simply denote all of them as $\pm$.

$$\begin{aligned}
\det A(1|1) &= \pm(a_{m+1,m+1} - x)\det A(1, m+1|1, m+1) + \sum_{k=2}^{s} \pm a_{km+1,m+1} \det A(1, km+1|1, m+1). \\
\det A(m+1|1) &= \pm a_{1,m+1} \det A(1, m+1|1, m+1) \pm a_{2m+1,m+1} \det A(m+1, 2m+1|1, m+1) \pm \cdots \\
&\quad \pm a_{sm+1,m+1} \det A(m+1, sm+1|1, m+1) \\
&= \sum_{\substack{k=0 \\ k\neq 1}}^{s} \pm a_{km+1,m+1} \det A(km+1, m+1|1, m+1).
\end{aligned}$$

$$\begin{aligned}
\det A(2m+1|1) &= \pm a_{1,m+1} \det A(1, 2m+1|1, m+1) \pm (a_{m+1,m+1} - x)\det A(m+1, 2m+1|1, m+1) \\
&\quad \pm a_{3m+1,m+1} \det A(2m+1, 3m+1|1, m+1) \pm \cdots \\
&\quad \pm a_{sm+1,m+1} \det A(2m+1, sm+1|1, m+1) \\
&= \sum_{\substack{k=0 \\ k\neq 2}}^{s} \pm a_{km+1,m+1} \det A(2m+1, km+1|1, m+1) \mp x \det A(m+1, 2m+1|1, m+1) \\
&\ \ \vdots
\end{aligned}$$

$$\begin{aligned}\det A(sm+1|1) &= \pm a_{1,m+1}\det A(1,sm+1|1,m+1) \pm (a_{m+1,m+1}-x)\det A(m+1,sm+1|1,m+1)\\ &\quad \pm a_{2m+1,m+1}\det A(2m+1,sm+1|1,m+1) \pm \cdots\\ &\quad \pm a_{(s-1)m+1,m+1}\det A((s-1)m+1,sm+1|1,m+1)\\ &= \sum_{\substack{k=0\\ k\neq s}}^{s} \pm a_{km+1,m+1}\det A(sm+1,km+1|1,m+1) \mp x\det A(m+1,sm+1|1,m+1)\end{aligned}$$

We need to take stock of what is happening. Each of these minors has the form $A(*|1,m+1)$. It is clear and we continue to evaluate the determinants of these minors, the next set will have the form $A(*|1,m+1,2m+1)$ and after r_0 iterations will have the form $A(*|1,m+1,2m+1,\ldots,sm+1)$.

Also at our current stage of computation, all minors of the form $A(jm+1,km+1|1,m+1)$ where $j\neq k\in\{0,\ldots,s\}$ also occur. At each new stage a new row will be added to the minor $jm+1,km+1,\ell m+1$ where j,k,l range over $0,\ldots,s$ with all indices distinct. After r_0 iterations, all r_0 rows $km+1$, $k=0,\ldots,s$ will necessarily appear in each minor, at which point we will have

$$\overline{\chi}_{1\otimes\gamma} = \overline{h}_1\cdot\det A(1,m+1,\ldots,sm+1|1,m+1,\ldots,sm+1).$$

Moreover, if A_0 was the image of the matrix of $1\otimes\gamma$ in $M_{r_{\mathfrak{P}}m_\nu}(\overline{\mathcal{O}}_{W_\nu})$ (so that the matrix A above is $A=\det(xI-A_0))$, we would have that

$$\det -A(1,m+1,\ldots,sm+1|1,m+1,\ldots,sm+1) = det(xI-A_0(1,\ldots,sm+1|1,\ldots,sm+1)),$$

that is the characteristic polynomial of a matrix in $M_{r_{\mathfrak{P}}m_\nu-r_0}(\mathcal{O}_{W_\nu})$, and thus having degree $r_{\mathfrak{p}}m_\nu-r_0$. This establishes that $\overline{\chi}_{1\otimes\gamma}=\overline{h}_1\overline{h}_2$ where $\deg\overline{h}_1=r_0<r_{\mathfrak{P}}$, which completes the proof. □

3. Constructing Distinguished Representatives of the Isomorphism Classes of Maximal Orders

The goal of this section is to use the local result (Theorem 2.1) and a local-global principle to construct a set of representatives of the isomorphism classes of maximal orders in B, and distinguish those which are guaranteed to contain $\mathcal{O}_L$. This task involves a number of steps. The first is to define a class field $K(\mathcal{R})/K$ whose degree is the number of isomorphism classes comprising the genus of $\mathcal{R}$. Then places ν of K are chosen so that the Artin symbols $(\nu,K(\mathcal{R})/K)$ correspond to generators of $\mathrm{Gal}(K(\mathcal{R})/K)$ in which ν has prescribed splitting behavior in L. Finally, a set of maximal orders in B are constructed by choosing distinguished representatives of the local algebras B_ν using Theorem 2.1. This broad outline was also followed in the simpler case of prime degree [20], but we include all the details here to afford careful treatment especially to the complications which arise due to the presence of partial ramification for central simple algebras of arbitrary degree.

3.1. **Class fields and the genus of $\mathcal{R}$.** First, we construct a class field, $K(\mathcal{R})$, associated to the maximal order $\mathcal{R}$ whose degree over K equals the number of isomorphism classes of maximal orders in the global algebra B. We then we give a filtration of the Galois group, $\mathrm{Gal}(K(\mathcal{R})/K)$, in order to parametrize the isomorphism classes of maximal orders in B.

The class field extension $K(\mathcal{R})/K$ comes from class field theory by producing an open subgroup $H_{\mathcal{R}}$ of finite index in the idele group J_K. The group $H_{\mathcal{R}}$ is the product of $K^\times$ and the reduced norm of an idelic normalizer of $\mathcal{R}$ ($nr(\mathfrak{N}(\mathcal{R}))$, where $\mathfrak{N}(\mathcal{R}) = J_B \cap \prod_\nu \mathcal{N}(\mathcal{R}_\nu)$, and where $\mathcal{N}(\mathcal{R}_\nu)$ is the local normalizer of $\mathcal{R}_\nu$ in $B_\nu^\times$, and J_B is the idele group of B.) We begin by computing the local normalizers and their reduced norms.

3.1.1. *Normalizers and their reduced norms.* Given our maximal order $\mathcal{R} \subset B$ and a place ν of K, we have previously defined the completions $\mathcal{R}_\nu \subseteq B_\nu$. Let $\mathcal{N}(\mathcal{R}_\nu)$ denote the normalizer of $\mathcal{R}_\nu$ in $B_\nu^\times$, and $nr_{B_\nu/K_\nu}(\mathcal{N}(\mathcal{R}_\nu))$ its reduced norm in $K_\nu^\times$. First suppose that ν is an infinite place, so $\mathcal{N}(\mathcal{R}_\nu) = B_\nu^\times$. If ν splits in B, then $B_\nu \cong \mathrm{M}_n(K_\nu)$, so $\mathcal{N}(R_\nu) \cong GL_n(K_\nu)$, and $nr_{B_\nu/K_\nu}(\mathcal{N}(\mathcal{R}_\nu)) = K_\nu^\times$, while if ν ramifies in B (possible only if n is even and ν is real), then (33.4) of [25] shows that $nr_{B_\nu/K_\nu}(\mathcal{N}(\mathcal{R}_\nu)) = \mathbb{R}_+^\times$.

For a finite place ν, it is clearest to distinguish three cases. If $m_\nu = 1$ (the split case), then B_ν has been identified with $\mathrm{M}_n(K_\nu)$, so by (17.3) and (37.26) of [25], every maximal order is conjugate by an element of $B_\nu^\times$ to $\mathrm{M}_n(\mathcal{O}_\nu)$, and every normalizer is conjugate to $\mathrm{GL}_n(\mathcal{O}_\nu)K_\nu^\times$, hence $nr_{B_\nu/K_\nu}(\mathcal{N}(\mathcal{R}_\nu)) = \mathcal{O}_\nu^\times (K_\nu^\times)^n$.

At the other extreme is $m_\nu = n$ (the totally ramified case), so that $B_\nu = D_\nu$. Then $\mathcal{R}_\nu$ is the unique maximal order of the division algebra B_ν, so $\mathcal{N}(\mathcal{R}_\nu) = B_\nu^\times$, and by p 153 of [25], $nr(\mathcal{N}(\mathcal{R}_\nu)) = nr_{B_\nu/K_\nu}(B_\nu^\times) = K_\nu^\times$.

Finally, consider the partially ramified case in which $B_\nu \cong \mathrm{M}_{r_\nu}(D_\nu)$ where D_ν is a central division algebra of degree $1 < m_\nu < n$ over K_ν. Then $\mathcal{R}_\nu$ is conjugate to $\mathrm{M}_{r_\nu}(\Delta_\nu)$ where Δ_ν is the unique maximal order of D_ν (17.3 of [25]).

From §14.5 of [25], we choose a uniformizer $\boldsymbol{\pi} = \boldsymbol{\pi}_{D_\nu}$ for Δ_ν so that $\boldsymbol{\pi}^{m_\nu} = \pi_\nu \in K_\nu$. We also take ω a primitive $(q^{m_\nu} - 1)$th root of unity in Δ_ν. Then $E_\nu = K_\nu(\boldsymbol{\pi})$ and $W_\nu = K_\nu(\omega)$ are degree m_ν field extensions of K_ν which are respectively totally ramified and unramifed and so that

$$\Delta_\nu = \mathcal{O}_\nu[\omega, \boldsymbol{\pi}] = \bigoplus_{i,j=0}^{m_\nu - 1} \mathcal{O}_\nu \omega^i \boldsymbol{\pi}^j \text{ and } D_\nu = K_\nu[\omega, \boldsymbol{\pi}]. \tag{9}$$

To deduce $nr(\mathcal{N}(\mathcal{R}_\nu))$, it is sufficient to consider $\mathcal{R}_\nu = M_{r_\nu}(\Delta_\nu)$. From (37.25)-(37.27) of [25], we know that $\mathcal{N}(\mathcal{R}_\nu)/GL_{r_\nu}(\Delta_\nu)K_\nu^\times \cong \mathbb{Z}/m_\nu\mathbb{Z}$. By (17.3) of [25], we know that $\boldsymbol{\pi}\mathcal{R}_\nu$ is the unique two-sided ideal of $\mathcal{R}_\nu$, which is to say that $\boldsymbol{\pi} \in \mathcal{N}(\mathcal{R}_\nu)$. It follows that $\mathcal{N}(\mathcal{R}_\nu)$ is the group generated by $\boldsymbol{\pi} I_{r_\nu}$ and $GL_{r_\nu}(\Delta_\nu)K_\nu^\times$. Since $nr_{D_\nu/K_\nu}(\boldsymbol{\pi}) = (-1)^{m_\nu - 1}\pi_\nu$, we have $nr_{B_\nu/K_\nu}(\boldsymbol{\pi} I_{r_\nu}) = (-1)^{r_\nu(m_\nu - 1)}\pi_\nu^{r_\nu}$. Finally, given that the unramified extension W_ν/K_ν

is contained in Δ_ν and and the norm N_{W_ν/K_ν} maps the units of $\mathcal{O}_{W_\nu}$ onto $\mathcal{O}_\nu^\times$, we may conclude that $nr(\mathcal{N}(\mathcal{R}_\nu)) = \mathcal{O}_\nu^\times (K_\nu^\times)^{r_\nu}$.

Summarizing, for a finite place ν of K, the computations above show that

$$nr(\mathcal{N}(\mathcal{R}_\nu)) = nr_{B_\nu/K_\nu}(\mathcal{N}(\mathcal{R}_\nu)) = \mathcal{O}_\nu^\times (K_\nu^\times)^{r_\nu},$$

for all $1 \leq r_\nu \leq n$.

Thus with the exception of a real place ν which ramifies in B (possible only if n is even), for all places ν we have $\mathcal{O}_\nu^\times \subset nr(\mathcal{N}(\mathcal{R}_\nu))$, a fact that will be important in associating a class field to $\mathcal{R}$.

3.1.2. *Parametrizing the Genus.* We know that any two maximal orders in B are locally conjugate at all (finite) places of K, so the number of isomorphism classes can be computed adelically as follows. Let J_B be the idele group of B, and let $\mathfrak{N}(\mathcal{R}) = J_B \cap \prod_\nu \mathcal{N}(\mathcal{R}_\nu)$ be the adelic normalizer of $\mathcal{R}$. The number of isomorphism classes of maximal orders is the cardinality of the double coset space $B^\times \backslash J_B / \mathfrak{N}(\mathcal{R})$. To make use of class field theory, we need to realize this quotient in terms of the arithmetic of K. The reduced norms on the local algebras B_ν induce a natural map $nr : J_B \to J_K$, where J_K is the idele group of K, and where for $\tilde{\alpha} = (\alpha_\nu)_\nu \in J_B$, $nr(\tilde{\alpha}) := (nr_{B_\nu/K_\nu}(\alpha_\nu))_\nu$.

The theorem below was proven (Theorem 3.1 of [20]) for $\deg_K B = p$ an odd prime. The changes required for general degree n involve handling possible ramification at an infinite place, and pervade the proof, so we repeat the full argument in the interest of clarity.

Theorem 3.1. *Let $n = \deg_K B \geq 3$. The reduced norm induces a bijection*

$$nr : B^\times \backslash J_B / \mathfrak{N}(\mathcal{R}) \to K^\times \backslash J_K / nr(\mathfrak{N}(\mathcal{R})).$$

The group $K^\times \backslash J_K / nr(\mathfrak{N}(\mathcal{R}))$ is abelian with exponent n.

Remark 3.2. The proof below is valid for $n = 2$ as well as long as B satisfies the Eichler condition. The map is always surjective, but injectivity requires strong approximation.

Proof. The map is defined in the obvious way with $nr(B^\times \tilde{\alpha} \mathfrak{N}(\mathcal{R})) = K^\times nr(\tilde{\alpha})\, nr(\mathfrak{N}(\mathcal{R}))$,

We first show the mapping is surjective. Let $\tilde{a} = (a_\nu)_\nu \in J_K$ and $K^\times \tilde{a}\, nr(\mathfrak{N}(\mathcal{R}))$ be the associated double coset in $K^\times \backslash J_K / nr(\mathfrak{N}(\mathcal{R}))$. The weak approximation theorem implies the existence of an element $c \in K^\times$ so that $c\tilde{a}$ satisfies $ca_\nu > 0$ for all real places ν of K which ramify in B (if any). Since (replacing a by ca) the associated double cosets are equal, we may assume without loss that $\tilde{a}$ was chosen with $a_\nu > 0$ are all the real places which ramify in B.

Now we appeal to (33.4) of [25] which says that for any place ν of K, $nr_{B_\nu/K_\nu}(B_\nu) = K_\nu$ with the sole exception of $K_\nu \cong \mathbb{R}$ and B ramified at ν in which case the image of the norm is the non-negative reals. Let S be a finite set of places of K containing all the archimedean places and all places which ramify in B. By (33.4) and the assumptions on $\tilde{a}$ at the real places, for each place $\nu \in S$, there exists $\beta_\nu \in B_\nu^\times$ so that $nr_{B_\nu/K_\nu}(\beta_\nu) = a_\nu$.

Now let ν be a place of K, with $\nu \notin S$. We have that $\mathcal{R}_\nu$ is conjugate to $M_n(\mathcal{O}_{K_\nu})$, so let $\beta_\nu \in \mathcal{R}_\nu$ be conjugate to $\mathrm{diag}(a_\nu, 1, \ldots, 1) \in M_n(\mathcal{O}_{K_\nu})$. Then $nr_{B_\nu/K_\nu}(\beta_\nu) = nr_{B_\nu/K_\nu}(\mathrm{diag}(a_\nu, 1, \ldots, 1)) = a_\nu$. So now put $\tilde{\beta} = (\beta_\nu)_\nu$. It is clear that $\tilde{\beta} = (\beta_\nu)_\nu \in J_B$ and $nr_{J_B/J_K}(\tilde{\beta}) = \tilde{a}$, which establishes surjectivity.

To prove injectivity, we first prove a claim: The preimage of $K^\times nr(\mathfrak{N}(\mathcal{R}))$ under nr is $B^\times J_B^1 \mathfrak{N}(\mathcal{R})$ where J_B^1 is the kernel of the norm map: $nr : J_B \to J_K$. It is obvious that $nr(B^\times J_B^1 \mathfrak{N}(\mathcal{R})) \subset K^\times nr(\mathfrak{N}(\mathcal{R}))$. Let $\tilde{\gamma} = (\gamma_\nu)_\nu \in J_B$ be such that $nr(B^\times \tilde{\gamma} \mathfrak{N}(\mathcal{R})) \in K^\times nr(\mathfrak{N}(\mathcal{R}))$. Then $nr(\tilde{\gamma}) \in K^\times nr(\mathfrak{N}(\mathcal{R}))$, so write $nr(\tilde{\gamma}) = a \cdot nr(\tilde{r})$ where $a \in K^\times$ and $\tilde{r} = (r_\nu)_\nu \in \mathfrak{N}(\mathcal{R})$. We claim that a is positive at all the real places which ramify in B. Indeed writing a_ν for the image of a under the embedding $K \subset K_\nu \cong \mathbb{R}$, we have that $nr_{B_\nu/K_\nu}(\gamma_\nu) = a_\nu nr_{B_\nu/K_\nu}(r_\nu)$, with $nr_{B_\nu/K_\nu}(\gamma_\nu), nr_{B_\nu/K_\nu}(r_\nu) > 0$. It follows by the Hasse-Schilling-Maass theorem (Theorem 33.15 of [25]) that there is an element $b \in B^\times$ so that $nr_{B/K}(b) = a$, and so that $nr(\tilde{\gamma}) = nr(b) nr(\tilde{r})$, or $nr(b^{-1}) nr(\tilde{\gamma}) nr(\tilde{\gamma}^{-1}) = 1 \in J_K$. Thus $b^{-1}\tilde{\gamma}\,\tilde{r}^{-1} \in J_B^1$, and $B^\times \tilde{\gamma} \mathfrak{N}(\mathcal{R}) = B^\times b^{-1} \tilde{\gamma}\,\tilde{r}^{-1} \mathfrak{N}(\mathcal{R}) \in B^\times J_B^1 \mathfrak{N}(\mathcal{R})$ as claimed.

To proceed with the proof of injectivity, suppose that that are $\tilde{\alpha}$, $\tilde{\beta} \in J_B$ so that $nr(B^\times \tilde{\alpha}\, nr(\mathfrak{N}(\mathcal{R})) = nr(B^\times \tilde{\beta}\, nr(\mathfrak{N}(\mathcal{R}))$. Then

$$K^\times nr(\tilde{\alpha}) nr(\mathfrak{N}(\mathcal{R})) = K^\times nr(\tilde{\beta}) nr(\mathfrak{N}(\mathcal{R})),$$

which since J_K is abelian, implies that $nr(\tilde{\alpha}^{-1}\tilde{\beta}) \in K^\times nr(\mathfrak{N}(\mathcal{R}))$, so by the above claim, $\tilde{\alpha}^{-1}\tilde{\beta} \in B^\times J_B^1 \mathfrak{N}(\mathcal{R})$.

Now the subgroup $B^\times J_B^1$ is the kernel of the homomorphism $J_B \to J_K/K^\times$ induced by nr, so that $\tilde{\beta} \in B^\times J_B^1 \mathfrak{N}(\mathcal{R}) = B^\times J_B^1 \tilde{\alpha} \mathfrak{N}(\mathcal{R})$. By VI.iii and VII of [15], $J_B^1 \subset B^\times \tilde{\gamma} \mathfrak{N}(\mathcal{R}) \tilde{\gamma}^{-1}$ for any $\tilde{\gamma} \in J_B$, so choosing $\tilde{\gamma} = \tilde{\alpha}$, we get

$$\tilde{\beta} \in B^\times J_B^1 \tilde{\alpha}\, \mathfrak{N}(\mathcal{R}) \subset B^\times (B^\times \tilde{\alpha}\, \mathfrak{N}(\mathcal{R})\, \tilde{\alpha}^{-1}) \tilde{\alpha}\, \mathfrak{N}(\mathcal{R}) = B^\times \tilde{\alpha}\, \mathfrak{N}(\mathcal{R}).$$

Thus $B^\times \tilde{\beta}\, \mathfrak{N}(\mathcal{R}) \subseteq B^\times \tilde{\alpha}\, \mathfrak{N}(\mathcal{R})$, and and by symmetry, we have equality.

To see that the group has exponent n, we note that the local factors in $J_K/nr(\mathfrak{N}(\mathcal{R}))$ have the form $K_\nu^\times / nr_{B_\nu/K_\nu}(\mathcal{N}(\mathcal{R}_\nu))$. From our computations above, we see that for ν a finite place, this quotient is either trivial or equal to $K_\nu^\times / (\mathcal{O}_\nu^\times (K_\nu^\times)^r)$ (for $r \mid n$) which clearly has exponent n, and that if ν is an infinite place, the quotient is trivial unless ν is a real place which ramifies in B. In that case, $K_\nu^\times / nr(\mathcal{N}(\mathcal{R}_\nu)) = \mathbb{R}^\times / \mathbb{R}_+^\times \cong \mathbb{Z}/2\mathbb{Z}$, but in that case n is necessarily even, so again the factor has exponent n. □

We have seen above that the distinct isomorphism classes of maximal orders in B are in one-to-one correspondence with the double cosets in the group $K^\times \backslash J_K / nr(\mathfrak{N}(\mathcal{R})) \cong G_{\mathcal{R}} := J_K / H_{\mathcal{R}}$, where $H_{\mathcal{R}} = K^\times nr(\mathfrak{N}(\mathcal{R}))$. Since $H_{\mathcal{R}}$ contains a neighborhood of the identity in J_K, it is an open subgroup (Proposition II.6 of [17]) having finite index, and so by class field theory [18], there is a class field, $K(\mathcal{R})$, associated to it. The extension $K(\mathcal{R})/K$ is an abelian extension with $\mathrm{Gal}(K(\mathcal{R})/K) \cong G_{\mathcal{R}}$. Moreover, a place ν of K (possibly infinite) is unramified in $K(\mathcal{R})$ if and only if $\mathcal{O}_\nu^\times \subset H_\mathcal{R}$, and splits completely if and only if $K_\nu^\times \subset H_\mathcal{R}$. Here if ν is archimedean, we take $\mathcal{O}_\nu^\times = K_\nu^\times$.

Remark 3.3. From our computations above, we see (unless there is a real place of K which ramifies in B) that $\mathcal{O}_\nu^\times$ is always contained in $H_\mathcal{R}$. In particular the class field $K(\mathcal{R})/K$ is unramified outside of the real places which ramify in B, so contained in the narrow class field of K.

It is also useful to make a simple observation about the order of Artin symbols in the class field extension $K(\mathcal{R})/K$. For a finite place ν of K and π_ν a uniformizer in K_ν, the isomorphism $G_\mathcal{R} = J_K/H_\mathcal{R} \to \mathrm{Gal}(K(R)/K)$ associates the image of the idele $\tilde{\omega}_\nu = (\dots, 1, \pi_\nu, 1, \dots)$ in $G_\mathcal{R}$ with the Artin symbol $(\nu, K(\mathcal{R})/K)$. Since $\tilde{\omega}_\nu^{r_\nu} = 1$ in $G_\mathcal{R}$ we have that the order of the Artin symbol (the inertial degree) $f(\nu; K(\mathcal{R})/K)$ divides r_ν. □

Our goal in what follows is to determine a subgroup H of the Galois group $G = \mathrm{Gal}(K(\mathcal{R})/K)$ so that each isomorphism class of maximal order in B corresponding to an element of H contains a representative which contains the ring of integers $\mathcal{O}_L$. On the other hand, the process of identifying the representatives containing $\mathcal{O}_L$ requires a slightly finer filtration of the group G which we establish below.

We begin by specifying a set of generators for the group G as Artin symbols, $(\nu, K(\mathcal{R})/K)$, in such a way that we can control the splitting behavior of ν in the extension L/K. As L is an arbitrary extension of K of degree n, this requires some care.

We have assumed that $L \subset B$. Put $L_0 = K(\mathcal{R}) \cap L$ and $\widehat{L}_0 = \widehat{L} \cap K(\mathcal{R})$ where $\widehat{L}$ is the Galois closure of L. Then $L_0 \subset \widehat{L}_0$ and we define subgroups of G: $\widehat{H} = \mathrm{Gal}(K(\mathcal{R})/\widehat{L}_0) \subseteq H = \mathrm{Gal}(K(\mathcal{R})/L_0)$. We write the finite abelian groups $\widehat{H}$, $H/\widehat{H}$, and G/H as a direct product of cyclic groups:

$$G/H = \langle \rho_1 H\rangle \times \cdots \times \langle \rho_r H\rangle, \tag{10}$$

$$H/\widehat{H} = \langle \sigma_1 \widehat{H}\rangle \times \cdots \times \langle \sigma_s \widehat{H}\rangle, \tag{11}$$

$$\widehat{H} = \langle \tau_1\rangle \times \cdots \times \langle \tau_t\rangle. \tag{12}$$

The following proposition is clear.

Proposition 3.4. *Every element $\varphi \in G$ can be written uniquely as $\varphi = \rho_1^{a_1} \cdots \rho_r^{a_r} \sigma_1^{b_1} \cdots \sigma_s^{b_s} \tau_1^{c_1} \cdots \tau_t^{c_t}$ where $0 \le a_i < |\rho_i H|$, $0 \le b_j < |\sigma_j \widehat{H}|$, and $0 \le c_k < |\tau_k|$, with $|\cdot|$ the order of the element in the respective group.*

Next we characterize each of these generators in terms of Artin symbols. Since the vehicle to accomplish this is the Chebotarev density theorem which provides an infinite number of choices for places, we may and do assume without loss that the places we choose to define the Artin symbols are unramified in both $\widehat{L}/K$ and B.

First consider the elements $\tau_k \in \widehat{H} = \mathrm{Gal}(K(\mathcal{R})/\widehat{L}_0)$. By Lemma 7.14 of [23], there exist infinitely many places ν_k of K so that $\tau_k = (\nu_k, K(\mathcal{R})/K)$ and for which there exists a place Q_k of $\widehat{L}$ with inertia degree $f(Q_k \mid \nu_k) = 1$. Since $\widehat{L}/K$ is Galois (and the place ν_k is unramified by assumption), this implies ν_k splits completely in $\widehat{L}$, hence also in L.

Next consider $\sigma_j \widehat{H}$ with $\sigma_j \in H = \mathrm{Gal}(K(\mathcal{R})/L_0)$. Again by Lemma 7.14 of [23], there exist infinitely many places μ_j of K so that $\sigma_j = (\mu_j, K(\mathcal{R})/K)$ and for which there exists a place Q_j of L with inertia degree $f(Q_j \mid \mu_j) = 1$. Here the μ_j need not split completely in L.

Finally consider $\rho_k H$ with $\rho_k \in G = \mathrm{Gal}(K(\mathcal{R})/K)$. By Chebotarev, there exist infinitely many places λ_i of K so that $\rho_i = (\lambda_i, K(\mathcal{R})/K)$. For later convenience, we note that by standard properties of the Artin symbol, $\overline{\rho}_i = \rho_i|_{L_0} = (\lambda_i, L_0/K)$ whose order in $\mathrm{Gal}(L_0/K)$ is equal to the inertia degree $f(\lambda_i; L_0/K)$.

As we said above, we have assumed without loss that all the places λ_i, μ_j, ν_k are unramified in $\widehat{L}$ and not totally ramified in B.

3.2. **Fixing representatives of the isomorphism classes.** In the previous subsection, we have chosen generators for $\mathrm{Gal}(K(\mathcal{R})/K)$ which are characterized as Artin symbols, in particular associated to certain finite places of K whose splitting behavior in our given extension L/K is somewhat controlled. We recall that the size of the Galois group equals the number of isomorphism classes of maximal orders in B. At each of those places ν associated to an Artin symbol, we consider the local algebra, B_ν, and specify a certain collection of maximal orders in it (the number being equal to the order of the Artin symbol $(\nu, K(\mathcal{R})/K)$), and loosely speaking, take as many local orders as possible which contain $\mathcal{O}_L$. We will then fix representatives of the isomorphism classes of maximal orders in B by utilizing a local-global correspondence.

As above, $\mathcal{R}$ is a fixed maximal order of B containing $\mathcal{O}_L$. For a finite place ν of K which is not totally ramified in B, we have $B_\nu \cong M_{r_\nu}(D_\nu)$, with D_ν a central division algebra over K_ν with unique maximal order Δ_ν, and $r_\nu > 1$. We fix an apartment in the affine building for $SL_{r_\nu}(D_\nu)$ which contains the vertex corresponding to the maximal order $\mathcal{R}_\nu$. We may select a basis $\{\alpha_1, \dots, \alpha_{r_\nu}\}$ of $D_\nu^{r_\nu}$ so that $\mathcal{R}_\nu = M_{r_\nu}(\Delta_\nu) \cong \mathrm{End}_{\Delta_\nu}(\Lambda)$ where $\Lambda = \bigoplus_{i=1}^{r_\nu} \Delta_\nu \alpha_i$. With $\boldsymbol{\pi}$ a uniformizer of Δ_ν, the vertices of the apartment are in bijective correspondence with those maximal orders of B_ν which are given as endomorphism rings of lattices of the form $\bigoplus_{i=1}^{r_\nu} \Delta_\nu \boldsymbol{\pi}^{a_i} \alpha_i$, the homothety class of which we abbreviate by $[a_1, \dots, a_{r_\nu}] \in \mathbb{Z}^{r_\nu}/\mathbb{Z}(1, \dots, 1)$. We shall identify the vertices of the apartment with these homothety classes of lattices.

Let's understand how this applies to choosing our representatives for the isomorphism classes. Since $L \subset B$, we know (by the Albert-Brauer-Hasse-Noether theorem) that $m_\nu \mid [L_{\mathfrak{P}} : K_\nu]$ for all places ν of K and places $\mathfrak{P}$ of L lying above ν. For finite places ν, we have that $L_{\mathfrak{P}}$ embeds as a K_ν-algebra into $B_\nu \cong M_{r_{\mathfrak{P}}}(D_\nu)$ where $r_{\mathfrak{P}} = [L_{\mathfrak{P}} : K_\nu]/m_\nu$ is minimal.

Corresponding to various generators of the $\mathrm{Gal}(K(\mathcal{R})/K)$ we have chosen finite places λ_i, μ_j, and ν_k to parametrize the Artin symbols which represent the generators. We now consider maximal orders in the associated local algebras. For a generic place ν among these (which we recall can be assumed unramified in L/K), let $\nu\mathcal{O}_L = \mathfrak{P}_1 \cdots \mathfrak{P}_g$ be the prime factorization in L. By Theorem 2.1, we know that $\mathcal{O}_L$ is a subset of precisely those maximal orders (vertices of the apartment) associated to homothety classes of lattices of the form $[\mathcal{L}] = [\underbrace{\ell_1, \dots, \ell_1}_{r_{\mathfrak{P}_1}}, \underbrace{\ell_2, \dots, \ell_2}_{r_{\mathfrak{P}_2}}, \dots, \underbrace{\ell_g, \dots, \ell_g}_{r_{\mathfrak{P}_g}}]$, $\ell_i \in \mathbb{Z}$.

We will be particularly interested in maximal orders of the form $\mathcal{R}(k, \ell)$ defined in equation (3). Because we shall vary the place ν in the parametrization below, we will write $\mathcal{R}_\nu(k, \ell)$ for $\mathcal{R}(k, \ell)$ to make the dependence on ν explicit. Recall that $\mathcal{R}_\nu(k, \ell)$ corresponds to the homothety class $[\underbrace{\ell, \dots, \ell}_{k}, 0, \dots, 0] \in \mathbb{Z}^{r_\nu}/\mathbb{Z}(1, \dots, 1)$ which has type $k\ell \pmod{r_\nu}$.

By equation (5),

$$\mathcal{O}_L \subset \bigcap_{\ell_i \in \mathbb{Z}} \left[\mathcal{R}_\nu(r_{\mathfrak{P}_1}, \ell_1) \cap \mathcal{R}_\nu(r_{\mathfrak{P}_1} + r_{\mathfrak{P}_2}, \ell_2) \cap \cdots \cap \mathcal{R}_\nu(r_{\mathfrak{P}_1} + \cdots + r_{\mathfrak{P}_g}, \ell_g)\right] = \bigoplus_{\mathfrak{P}|\nu} M_{r_{\mathfrak{P}}}(\Delta_\nu) \subset M_{r_\nu}(D_\nu).$$

Now for the places λ_i, μ_j, and ν_k we specified above to parametrize $G = \mathrm{Gal}(K(\mathcal{R})/K)$, fix the following local orders using the decomposition of G into G/H, $H/\widehat{H}$, and $\widehat{H}$:

The places ν_k all split completely in L, so $L_{\mathfrak{P}} = K_{\nu_k}$, and $m_\nu \mid [L_{\mathfrak{P}} : K_{\nu_k}]$ implies $r_{\mathfrak{P}} = m_{\nu_k} = 1$, and that $r_{\nu_k} = n$.

So for each place ν_k ($k = 1, \dots, t$) whose Artin symbol $(\nu_k, K(\mathcal{R})/K)) = \tau_k$ is one of the generators of $\widehat{H}$, we fix vertices $\mathcal{R}_{\nu_k}(m, 1)$, $m = 0, 1, \dots, |\tau_k| - 1$ with associated homothety classes $[0, \dots, 0]$, $[1, 0, \dots, 0]$, $[1, 1, 0, \dots, 0]$, $\dots$, $[\underbrace{1, \dots, 1}_{|\tau_k|-1}, 0, \dots, 0]$. Note that since $r_{\nu_k} = n$ and τ_k has exponent n, all these homothety classes correspond to vertices in a fundamental chamber of the building, and the corresponding maximal orders contain $\mathcal{O}_L$ by equation (5).

Now consider the places μ_j ($j = 1, \dots, s$) whose Artin symbol $(\mu_j, K(\mathcal{R})/K)) = \sigma_j$ gives one of the generators $\sigma_j\widehat{H}$ of $H/\widehat{H}$. Recall that each μ_j factors into places of L with at least one having inertia degree one over μ_j, say $\mathfrak{P}_1$. Since μ_j is (by choice) unramified in L, we have as in the previous case $m_{\mu_j} \mid [L_{\mathfrak{P}_1} : K_{\mu_j}] = 1$, which forces $m_{\mu_j} = r_{\mathfrak{P}_1} = 1$ and $r_{\mu_j} = r_{\mu_j} m_{\mu_j} = n$. From equation (5), $\mathcal{O}_L \subset \mathcal{R}_{\mu_j}(r_{\mathfrak{P}_1}, \ell_1) = \mathcal{R}_{\mu_j}(1, \ell_1)$ for all $\ell_1 \in \mathbb{Z}$, so we fix vertices $\mathcal{R}_{\mu_j}(1, m)$, $m = 0, 1, \dots, |\sigma_j\widehat{H}| - 1$ with associated homothety classes $[0, \dots, 0]$, $[1, 0, \dots, 0]$, $[2, 0, \dots, 0]$, $\dots$, $[|\sigma_j\widehat{H}| - 1, 0, \dots, 0]$ in a fundamental apartment.

Finally consider the places λ_i ($i = 1, \dots, r$) whose Artin symbol $(\lambda_i, K(\mathcal{R})/K)) = \rho_i$ gives one of the generators $\rho_i H$ of G/H. It is only here where selectivity can manifest itself.

Recall that via the isomorphism $G/H \cong \mathrm{Gal}(L_0/K)$ ($\rho_i H \leftrightarrow \overline{\rho}_i$), we know that the order of $\rho_i H$ is the inertia degree $f(\lambda_i; L_0/K)$ which we have shown divides r_{λ_i}. So we wish to specify $f(\lambda_i; L_0/K)$ maximal orders in the local algebra. From Theorem 2.1, we know that $\mathcal{O}_L$ is contained in maximal orders corresponding precisely to vertices whose associated homothety classes are of the form $[\underbrace{\ell_1, \dots, \ell_1}_{r_{\mathfrak{P}_1}}, \underbrace{\ell_2, \dots, \ell_2}_{r_{\mathfrak{P}_2}}, \dots, \underbrace{\ell_g, \dots, \ell_g}_{r_{\mathfrak{P}_g}}]$, in particular having types $\sum_{k=1}^{g} r_{\mathfrak{P}_k} \ell_k \pmod{r_{\lambda_i}}$. Since the ℓ_k are arbitrary integers, $\mathcal{O}_L$ is contained in maximal orders having types which are multiples of $d_{\lambda_i} = \gcd(r_{\mathfrak{P}_1}, \dots, r_{\mathfrak{P}_g})$; note that $d_{\lambda_i} \mid r_{\lambda_i} = \sum_{k=1}^{g} r_{\mathfrak{P}_k}$.

Remark 3.5. We need to be a bit careful in leveraging the above observation. We have shown that $\mathcal{O}_L$ is contained in maximal orders having types a multiple of d_{λ_i}, but the converse is not necessarily true. For example, suppose $r_{\lambda_i} = \sum_{k=1}^{g} r_{\mathfrak{P}_k} = 1 + 2$, so that $\mathcal{O}_L$ is contained in maximal orders corresponding to homothety classes of the form $[\ell_1, \ell_2, \ell_2]$. Now $d_{\lambda_i} = 1$, so $\mathcal{O}_L$ is contained in maximal orders associated to homothety classes of all types, in particular type 1, but for example $\mathcal{O}_L$ is not contained in the maximal order corresponding to the homothety class of the lattice $[0, 1, 0]$ since that is not of the prescribed form: $[\ell_1, \ell_2, \ell_2]$. This presents no serious issue, but we need to be somewhat careful in selecting our representatives.

Fix integers $\ell_1, \dots, \ell_k$ so that

$$d_{\lambda_i} = \gcd(r_{\mathfrak{P}_1}, \dots, r_{\mathfrak{P}_g}) = r_{\mathfrak{P}_1} \ell_1 + \cdots + r_{\mathfrak{P}_g} \ell_g,$$

and fix a vertex corresponding to the homothety class

$$[\mathcal{L}] = [\underbrace{\ell_1, \dots, \ell_1}_{r_{\mathfrak{P}_1}}, \underbrace{\ell_2, \dots, \ell_2}_{r_{\mathfrak{P}_2}}, \dots, \underbrace{\ell_g, \dots, \ell_g}_{r_{\mathfrak{P}_g}}].$$

Using somewhat ad hoc notation, for an integer a, let

$$[\mathcal{L}^a] = [\underbrace{a\ell_1, \dots, a\ell_1}_{r_{\mathfrak{P}_1}}, \underbrace{a\ell_2, \dots, a\ell_2}_{r_{\mathfrak{P}_2}}, \dots, \underbrace{a\ell_g, \dots, a\ell_g}_{r_{\mathfrak{P}_g}}],$$

which has type $ad_{\lambda_i} \pmod{r_{\lambda_i}}$. Now

$$d_{\lambda_i} x \equiv d_{\lambda_i} y \pmod{r_{\lambda_i}} \text{ iff } x \equiv y \pmod{r_{\lambda_i}/d_{\lambda_i}},$$

so this process will produce $r_{\lambda_i}/d_{\lambda_i}$ maximal orders which contain $\mathcal{O}_L$, representing every possible type of maximal order which can contain $\mathcal{O}_L$. It turns out that in general, there will be some redundancy when we use these local orders to construct global ones via a local-global correspondence. We need to correct for this, and we begin with an elementary claim.

Lemma 3.6. *With the notation as above except abbreviating $f(\lambda_i; L_0/K)$ by f_{λ_i}, we have*

$$\frac{f_{\lambda_i}}{\gcd(d_{\lambda_i}, f_{\lambda_i})} \,\Big|\, \frac{r_{\lambda_i}}{d_{\lambda_i}}.$$

Proof. We know that $f_{\lambda_i} \mid r_{\lambda_i}$ and $d_{\lambda_i} \mid r_{\lambda_i}$. Then

$$\frac{r_{\lambda_i}}{d_{\lambda_i}} \cdot \frac{\gcd(d_{\lambda_i}, f_{\lambda_i})}{f_{\lambda_i}} = \frac{r_{\lambda_i}}{\operatorname{lcm}(d_{\lambda_i}, f_{\lambda_i})},$$

which is clearly integral. □

Above, we observed that types $d_{\lambda_i}x \equiv d_{\lambda_i}y \pmod{r_{\lambda_i}}$ iff $x \equiv y \pmod{r_{\lambda_i}/d_{\lambda_i}}$, so given the lemma, if we choose orders of types $d_{\lambda_i}x$ with x modulo $f_{\lambda_i}/\gcd(d_{\lambda_i}, f_{\lambda_i})$, they will be distinct modulo both r_{λ_i} and f_{λ_i}.

We want to fix maximal orders $\mathcal{R}_{\lambda_i}^m$ where $m \in \mathbb{Z}/f_{\lambda_i}\mathbb{Z}$; we separate those residues which can be written as $m \equiv d_{\lambda_i}a \pmod{f_{\lambda_i}}$ from those that cannot. We put $\mathcal{R}_{\lambda_i}^{d_{\lambda_i}a} := \operatorname{End}_{\Delta_{\lambda_i}}([\mathcal{L}^a])$ for $a = 0, 1, \ldots, f_{\lambda_i}/\gcd(d_{\lambda_i}, f_{\lambda_i}) - 1$, and for m one of the remaining $f_{\lambda_i} - f_{\lambda_i}/\gcd(d_{\lambda_i}, f_{\lambda_i})$ residues, choose a maximal order associated to a homothety class of lattice having type m. Recall that $f_{\lambda_i} \mid r_{\lambda_i}$, so these choices are possible.

Remark 3.7. We note from our remarks above, that $\mathcal{O}_L$ is a subset of $\mathcal{R}_{\mu_j}(1, m)$ for every value of m, and of $\mathcal{R}_{\nu_k}(m', 1)$ for $0 \le m' \le n$.

Now we use the local-global correspondence for orders to define global orders from the above local factors. Fix the following notation:

$$\begin{aligned}
\mathbf{a} &= (a_i) \in \mathbb{Z}/|\rho_1 H|\mathbb{Z} \times \cdots \times \mathbb{Z}/|\rho_r H|\mathbb{Z},\\
\mathbf{b} &= (b_j) \in \mathbb{Z}/|\sigma_1 \widehat{H}|\mathbb{Z} \times \cdots \times \mathbb{Z}/|\sigma_s \widehat{H}|\mathbb{Z},\\
\mathbf{c} &= (c_k) \in \mathbb{Z}/|\tau_1|\mathbb{Z} \times \cdots \times \mathbb{Z}/|\tau_t|\mathbb{Z}.
\end{aligned}$$

Here we assume the coordinates a_i, b_j, c_k are integers which are the least non-negative residues corresponding to the moduli. Define maximal orders, $\mathcal{D}^{\mathbf{a},\mathbf{b},\mathbf{c}}$, in B via the local-global correspondence:

$$\mathcal{D}_{\mathfrak{p}}^{\mathbf{a},\mathbf{b},\mathbf{c}} = \begin{cases} \mathcal{R}_{\mathfrak{p}} & \text{if } \mathfrak{p} \notin \{\lambda_i, \mu_j, \nu_k\},\\ \mathcal{R}_{\lambda_i}^{a_i} & \text{if } \mathfrak{p} = \lambda_i, i = 1, \ldots, r,\\ \mathcal{R}_{\mu_j}^{b_j} := \mathcal{R}_{\mu_j}(1, b_j) & \text{if } \mathfrak{p} = \mu_j, j = 1, \ldots, s,\\ \mathcal{R}_{\nu_k}^{c_k} := \mathcal{R}_{\nu_k}(c_k, 1) & \text{if } \mathfrak{p} = \nu_k, k = 1, \ldots, t. \end{cases} \tag{13}$$

We claim that such a collection of maximal orders parametrizes the isomorphism classes of maximal orders in B. That is, given any maximal order $\mathcal{E}$ in B, we show there are unique tuples $\mathbf{a}, \mathbf{b}, \mathbf{c}$ so that $\mathcal{E} \cong \mathcal{D}^{\mathbf{a},\mathbf{b},\mathbf{c}}$. To see this we again employ a local-global principle. We know that any two maximal orders in B are equal at almost all places of K, so they are distinguished at only a finite number of places. We collect information about those differences by defining a "distance idele" associated to the two orders.

Let $\mathfrak{M}$ denote the set of all maximal orders in B, and let $\mathcal{R}_1, \mathcal{R}_2 \in \mathfrak{M}$. For each place ν of K we want to define a local "type distance", $td_\nu(\mathcal{R}_{1\nu}, \mathcal{R}_{2\nu})$, which distinguishes the local orders. For infinite places ν, $\mathcal{R}_{1\nu} = \mathcal{R}_{2\nu} = B_\nu$, so (whatever the definition at other places) it makes sense to define $td_\nu(\mathcal{R}_{1\nu}, \mathcal{R}_{2\nu}) = 0$ in this case. We adopt the same convention for a finite place which totally ramifies in B, since there is a unique maximal order in B_ν. In the cases where a finite place splits or partially ramifies, we have already defined the type distance $td_\nu(\mathcal{R}_{1\nu}, \mathcal{R}_{2\nu})$ in section 2. In particular, td_ν is only well-defined modulo r_ν, but this causes no difficulty.

To return to the problem of parametrizing the isomorphism classes of maximal orders in B, we define a map (called the $G_\mathcal{R}$-valued distance idele) $\delta : \mathfrak{M} \times \mathfrak{M} \to G_\mathcal{R} = J_K/H_\mathcal{R}$ (where $H_\mathcal{R} = K^\times nr(\mathfrak{N}(\mathcal{R})))$ as follows: Given $\mathcal{R}_1$, $\mathcal{R}_2 \in \mathfrak{M}$, let $\delta(\mathcal{R}_1, \mathcal{R}_2)$ be the image in $G_\mathcal{R}$ of the idele $(\pi_\nu^{td_\nu(\mathcal{R}_{1\nu}, \mathcal{R}_{2\nu})})_\nu$, where π_ν is a fixed uniformizing parameter in K_ν (putting $\pi_\nu = 1$ at the archimedean places). Note that while the idele is not well-defined, its image in $G_\mathcal{R}$ is, since at any place where the type distance might be nontrivial, the local factor in $H_\mathcal{R}$ equals $\mathcal{O}_\nu^\times (K_\nu^\times)^{r_\nu}$.

That the orders $\{\mathcal{D}^{\mathbf{a},\mathbf{b},\mathbf{c}}\}$ parametrize the isomorphism classes of maximal orders in B follows from the the following proposition.

Proposition 3.8. *Let $\mathcal{R}$ be a fixed maximal order in B, and consider the collection of maximal orders $\mathcal{D}^{\mathbf{a},\mathbf{b},\mathbf{c}}$ defined above.*

(1) If $\mathcal{E}$ is a maximal order in B and $\mathcal{E} \cong \mathcal{R}$, then $\delta(\mathcal{R}, \mathcal{E})$ is trivial.
(2) If $\mathcal{E} \cong \mathcal{E}'$ are maximal orders in B, then $\delta(\mathcal{R}, \mathcal{E}) = \delta(\mathcal{R}, \mathcal{E}')$.
(3) $\mathcal{D}^{\mathbf{a},\mathbf{b},\mathbf{c}} \cong \mathcal{D}^{\mathbf{a}',\mathbf{b}',\mathbf{c}'}$ if and only if $\mathbf{a} = \mathbf{a}'$, $\mathbf{b} = \mathbf{b}'$, and $\mathbf{c} = \mathbf{c}'$.
(4) If $\mathcal{E}$ and $\mathcal{E}'$ are maximal orders in B, and $\delta(\mathcal{R}, \mathcal{E}) = \delta(\mathcal{R}, \mathcal{E}')$, then $\mathcal{E} \cong \mathcal{E}'$.

Proof. For the first assertion, we may assume that $\mathcal{E} = b\mathcal{R}b^{-1}$ for some $b \in B^\times$ by Skolem-Noether. Thus for each place ν, $\mathcal{E}_\nu = b\mathcal{R}_\nu b^{-1}$. The goal is to show that $\delta(\mathcal{R}, \mathcal{E}) = 1$ in $G_\mathcal{R}$, by showing that the distance idele which is derived from the local type distances is the same as the principal idele $(nr_{B/K}(b))$ which lies in the image of $K^\times$ in J_K.

We first verify that the local factors of the principal idele $(nr_{B/K}(b))$ are also trivial in $G_\mathcal{R}$. Indeed the local factors in $G_\mathcal{R}$ are trivial at both the infinite and totally ramified places with the possible exception of a real place which ramifies in B, but it follows from (33.4) of [25] that the norm is positive which is trivial in the local factor $\mathbb{R}^\times/\mathbb{R}_+^\times$.

Thus we need only consider places ν which are split or partially ramified in B. We handle these cases together as in our description above, and assume B_ν has been identified with $\mathrm{M}_{r_\nu}(D_\nu)$ where D_ν is a central division algebra of degree m_ν over K_ν. As before we let Δ_ν be the unique maximal order in D_ν. For convenience assume that the identification of B_ν with $\mathrm{M}_{r_\nu}(D_\nu)$ is done in such a way that, as described in the previous section, there is a rank r_ν free Δ_ν-lattice Λ_ν so that $\mathcal{R}_\nu = \mathrm{End}_{\Delta_\nu}(\Lambda_\nu)$, and hence $\mathcal{E}_\nu = \mathrm{End}_{\Delta_\nu}(b\Lambda_\nu)$ for some

$b \in \mathrm{GL}_{r_\nu}(D_\nu)$. Using elementary divisors for Δ_ν-lattices, we may assume without loss that $b = \mathrm{diag}(\boldsymbol{\pi}_{D_\nu}^{a_1}, \ldots, \boldsymbol{\pi}_{D_\nu}^{a_{r_\nu}})$. Then $td_\nu(\mathcal{R}_\nu, \mathcal{E}_\nu) \equiv \sum_{i=1}^{r_\nu} a_i \pmod{r_\nu}$.

So we shall compare the cosets $\pi_\nu^{\sum_{i=1}^{r_\nu} a_i}\mathcal{O}_\nu(K_\nu^\times)^{r_\nu}$ with $\pi_\nu^\ell \mathcal{O}_\nu(K_\nu^\times)^{r_\nu}$ where $\ell = \mathrm{ord}_{\pi_\nu}(nr_{B_\nu/K_\nu}(b))$.

We check that indeed $\ell \equiv \sum_{i=1}^{r_\nu} a_i \pmod{r_\nu}$ as follows. With $b = \mathrm{diag}(\boldsymbol{\pi}_{D_\nu}^{a_1}, \ldots, \boldsymbol{\pi}_{D_\nu}^{a_{r_\nu}}) \in B_\nu^\times = GL_{r_\nu}(D_\nu)$, we recall from earlier $nr_{D_\nu/K_\nu}(\boldsymbol{\pi}_{D_\nu}) = (-1)^{m_\nu - 1}\pi_\nu$, so (up to units in $\mathcal{O}_\nu$) $nr_{B_\nu/K_\nu}(b) = \pi_\nu^{\sum a_i}$, hence the result.

Thus we see that $\delta(\mathcal{R}, \mathcal{E})$ is the image in $G_\mathcal{R}$ of the principal idele $(nr_{B/K}(b))_\nu$, so $\delta(\mathcal{R}, \mathcal{E}) = 1$ in $G_\mathcal{R} = J_k/K^\times nr(\mathfrak{N}(\mathcal{R}))$ as $(nr_{B/K}(b))_\nu$ is in the image of $K^\times$ in J_K.

For the second claim, we may write $\mathcal{E}' = b\mathcal{E}b^{-1}$ for some $b \in B^\times$, so $\mathcal{E}'_\nu = b\mathcal{E}_\nu b^{-1}$ for each place ν, and as in the previous part, we need only worry about those places ν which split or are partially ramified in B. So as before, we write $\mathcal{R}_\nu = \mathrm{End}_{\Delta_\nu}(\Lambda_\nu)$ and $\mathcal{E}_\nu = \mathrm{End}_{\Delta_\nu}(\Gamma_\nu)$, so that $\mathcal{E}'_\nu = \mathrm{End}_{\Delta_\nu}(b\Gamma_\nu)$, where Λ_ν and Γ_ν are free Δ_ν-lattices of rank r_ν. Considering the invariant factors of the lattices Λ_ν, Γ_ν and $b\Gamma_\nu$, we easily see that

$$\delta(\mathcal{R}, \mathcal{E}') = \delta(\mathcal{R}, \mathcal{E})\delta(\mathcal{E}, \mathcal{E}') = \delta(\mathcal{R}, \mathcal{E}),$$

since $\delta(\mathcal{E}, \mathcal{E}') = 1$ by the computations in the first part.

For the third statement, we need only show one direction. Let ν be a finite place of K and π_ν the corresponding uniformizing parameter of K_ν. Let $\tilde{\omega}_\nu$ denote the idele with π_ν in the νth place and 1's elsewhere. Observe that Artin reciprocity identifies the image of $\tilde{\omega}_\nu$ in $G_R = J_K/H_\mathcal{R}$ with the Artin symbol $(\nu, K(\mathcal{R})/K) \in \mathrm{Gal}(K(\mathcal{R})/K)$. Moreover, for two maximal orders $\mathcal{E}, \mathcal{E}'$ of B, we see that $\delta(\mathcal{E}, \mathcal{E}')$ is equal to the image of $\prod_\nu \pi_\nu^{td_\nu(\mathcal{E}, \mathcal{E}')}$ in $G_\mathcal{R}$, and hence corresponds to a product of Artin symbols.

We recall that the orders $\mathcal{D}^{\mathbf{a},\mathbf{b},\mathbf{c}}$ differ from our fixed maximal order $\mathcal{R}$ only at finite places which were unramified in both L and B. At such a place ν, we identified B_ν with $\mathrm{M}_{r_\nu}(D_\nu)$ and our representative maximal orders were identified as endomorphism rings of homothety classes of lattices relative to some fixed basis $\{\alpha_i\}$ of $D_\nu^{r_\nu}$. Now referring to the conventions we adopted for the places λ_i, μ_j, ν_k whose associated Artin symbols were used to parametrize $\mathrm{Gal}(K(\mathcal{R})/K)$, we check that $\pmod n$,

$$td_\nu(\delta(\mathcal{D}^{\mathbf{a},\mathbf{b},\mathbf{c}}, \mathcal{D}^{\mathbf{a}',\mathbf{b}',\mathbf{c}'}) \equiv \begin{cases} a'_i - a_i & \text{for } \nu = \lambda_i, \\ b'_j - b_j & \text{for } \nu = \mu_j, \\ c'_k - c_k & \text{for } \nu = \nu_k. \end{cases}$$

It follows that

$$\delta(\mathcal{D}^{\mathbf{a},\mathbf{b},\mathbf{c}}, \mathcal{D}^{\mathbf{a}',\mathbf{b}',\mathbf{c}'}) \leftrightarrow \prod_{i=1}^{g} \rho_i^{a'_i - a_i} \prod_{j=1}^{s} \sigma_j^{b'_j - b_j} \prod_{k=1}^{t} \tau_k^{c'_k - c_k} \in \mathrm{Gal}(K(\mathcal{R})/K),$$

which is trivial if and only if $\mathbf{a} = \mathbf{a}'$, $\mathbf{b} = \mathbf{b}'$, and $\mathbf{c} = \mathbf{c}'$ by Proposition 3.4.

Finally for the last statement, let $\mathcal{E}$ and $\mathcal{E}'$ be maximal orders in B with $\delta(\mathcal{R},\mathcal{E}) = \delta(\mathcal{R},\mathcal{E}')$. Suppose to the contrary that $\mathcal{E} \ncong \mathcal{E}'$. Then $\mathcal{E} \cong \mathcal{D}^{\mathbf{a},\mathbf{b},\mathbf{c}}$, $\mathcal{E}' \cong \mathcal{D}^{\mathbf{a}',\mathbf{b}',\mathbf{c}'}$ where at least one of $\mathbf{a},\mathbf{b},\mathbf{c}$ differs from $\mathbf{a}',\mathbf{b}',\mathbf{c}'$. Since $\mathcal{R} = \mathcal{D}^{\mathbf{0},\mathbf{0},\mathbf{0}}$, the computations above show that $\delta(\mathcal{R},\mathcal{D}^{\mathbf{a},\mathbf{b},\mathbf{c}}) \neq \delta(\mathcal{R},\mathcal{D}^{\mathbf{a}',\mathbf{b}',\mathbf{c}'})$, but by part (2) of the proposition $\delta(\mathcal{R},\mathcal{E}) = \delta(\mathcal{R},\mathcal{D}^{\mathbf{a},\mathbf{b},\mathbf{c}})$, and $\delta(\mathcal{R},\mathcal{E}') = \delta(\mathcal{R},\mathcal{D}^{\mathbf{a}',\mathbf{b}',\mathbf{c}'})$, which provides the desired contradiction. This completes the proof. □

We now summarize our efforts in this section labeling those isomorphism classes of maximal orders in B which contain (a representative containing) the ring of integers $\mathcal{O}_L$. Above we have parametrized the isomorphism classes of maximal orders by the set $\{\mathcal{D}^{\mathbf{a},\mathbf{b},\mathbf{c}}\}$ given in equation (13). These orders are locally equal to $\mathcal{R}$ at all places except those designated previously as a member of the set $T = \{\lambda_1,\dots,\lambda_r,\mu_1,\dots,\mu_s,\nu_1,\dots\nu_t\}$. By this assumption, for $\mathfrak{p} \notin T$, we have $\mathcal{O}_L \subset \mathcal{D}_{\mathfrak{p}}^{\mathbf{a},\mathbf{b},\mathbf{c}}$. For $\mathfrak{p} = \mu_j$ or ν_k, we also have $\mathcal{O}_L \subset \mathcal{D}_{\mathfrak{p}}^{\mathbf{a},\mathbf{b},\mathbf{c}}$ by Remark 3.7. Finally, $\mathcal{O}_L \subset \mathcal{R}_{\lambda_i} = \mathcal{D}_{\lambda_i}^{\mathbf{0},\mathbf{b},\mathbf{c}}$ for all the places λ_i. Thus, for all finite $\mathfrak{p}$ in K, $\mathcal{O}_L \subset \mathcal{D}_{\mathfrak{p}}^{\mathbf{a},\mathbf{b},\mathbf{c}}$ for all $\mathbf{b},\mathbf{c}$, and $\mathbf{a} = \mathbf{0}$, and so by the local-global correspondence, $\mathcal{O}_L \subset \mathcal{D}^{\mathbf{0},\mathbf{b},\mathbf{c}}$ for all $\mathbf{b},\mathbf{c}$. But these orders $\{\mathcal{D}^{\mathbf{0},\mathbf{b},\mathbf{c}}\}$ are precisely those which correspond to the elements of $H = \mathrm{Gal}(K(\mathcal{R})/L_0)$. We summarize this as

Theorem 3.9. *The ring of integers, $\mathcal{O}_L$ is contained in at least $[K(\mathcal{R}) : L_0]$ of the $[K(\mathcal{R}) : K]$ representatives $\{\mathcal{D}^{\mathbf{a},\mathbf{b},\mathbf{c}}\}$. Specifically, $\mathcal{O}_L \subset \mathcal{D}^{\mathbf{0},\mathbf{b},\mathbf{c}}$ for all $\mathbf{b},\mathbf{c}$.*

4. Recovering global selectivity results

In this section we recover and refine some global results on selective orders. Recall that we have a central simple algebra $B = M_r(D)$ where D is a central division algebra of degree m over a number field K. We have an extension L/K of degree $n = rm$ which embeds in B, and we have fixed a maximal order $\mathcal{R}$ of B which contains $\mathcal{O}_L$. Associated to $\mathcal{R}$ is a class field, $K(\mathcal{R})$, and we have set $L_0 = K(\mathcal{R}) \cap L$. We assume $n \geq 3$.

4.1. **Simple lower bounds.** It is immediate from Theorem 3.9, that the ring of integers, $\mathcal{O}_L$ is contained in at least $[K(\mathcal{R}) : L_0]$ maximal orders which lie in distinct isomorphism classes, so speaking informally, at least $1/[L_0 : K]$ of the isomorphism classes "admit an embedding" of $\mathcal{O}_L$.

Having established $1/[L_0 : K]$ as a lower bound, we next show that the degree $[L_0 : K]$ is further constrained as a divisor of $[L : K] = n = rm$ in an interesting way.

Proposition 4.1. *Let $B = M_r(D)$ where D is a central division algebra of degree m over a number field K which contains an extension L/K of degree $n = rm$. Fix a maximal order $\mathcal{R}$ of B which contains the ring of integers $\mathcal{O}_L$. As above, associate to $\mathcal{R}$ a class field extension $K(\mathcal{R})/K$, and put $L_0 = L \cap K(\mathcal{R})$. If no real place of K ramifies in B, then $[L_0 : K] \mid r \cdot \gcd(r,m)$; otherwise $[L_0 : K] \mid 2r \cdot \gcd(r,m)$. In particular if $\gcd(r,m) = 1$, then $[L_0 : K] \mid r$ or $2r$.*

Remark 4.2. The proposition above extends the simplest form of Carmona's [4] result, where he shows that for an arbitrary division algebra, the selectivity proportion is $1/2$ or 1, which we see from above with $r = 1$.

Proof. This proof follows the lines of a similar argument in [4]. For each place ν of K, we have written $B_\nu \cong M_{r_\nu}(D_\nu)$ where D_ν is a central division algebra of degree m_ν over K_ν, and of course where $n = rm = r_\nu m_\nu$. By (32.17) of [25], we know that $m = \operatorname{lcm}\{m_\nu\}$ where the lcm is taken over all places of K.

To begin, let p be an odd prime, and assume $p^t \| m$, $t \geq 1$. Also assume that $p^s \| r$ with $s \geq 0$. Then there must be a place ν of K with $p^t \| m_\nu$. Since p is odd, we know ν is a finite place of K. Since L embeds in B, we know for every place $\mathfrak{P}$ lying above ν that

$$m_\nu \mid [L_\mathfrak{P} : K_\nu] = [L_\mathfrak{P} : (L_0)_{\mathfrak{P}\cap L_0}][(L_0)_{\mathfrak{P}\cap L_0} : K_\nu] = [L_\mathfrak{P} : (L_0)_{\mathfrak{P}\cap L_0}]f(\nu; L_0/K),$$

the last equality since $K(\mathcal{R})/K$ is abelian and unramified at all finite places. By Remark 3.3, we know that $f(\nu; K(\mathcal{R})/K) \mid r_\nu$, hence so does $f(\nu; L_0/K)$. Now since $n = rm = r_\nu m_\nu$ and $p^t \| m_\nu$ we have $p^s \| r_\nu$, so $\operatorname{ord}_p(f(\nu; L_0/K)) \leq s$. Let $t_0 = \max\{0, t - s\}$. Then $p^{t_0} \mid [L_\mathfrak{P} : (L_0)_{\mathfrak{P}\cap L_0}]$. It follows that $p^{t_0} \mid [L : L_0]$. Therefore

$$\operatorname{ord}_p[L_0 : K] \leq s + t - t_0 = \begin{cases} 2s & s \leq t \\ s + t & s > t \end{cases} = \operatorname{ord}_p(r) + \operatorname{ord}_p(\gcd(r, m)).$$

Which gives the result for the odd primes p. When $p = 2$, if $4 \mid m$, the same argument gives the correct bounds with $p = 2$. Moreover, even if $2 \| m$, but there is some finite place ν with $2 \mid m_\nu$, the argument is valid. It is only in the case that $2 \| m$, but for no finite place does $2 \mid m_\nu$ that the argument fails, and in that case we must have a real place which ramifies in B. ∎

4.2. **The effect of ramification on the bounds.** The ramification of the central simple algebra B has an interesting impact on selectivity. In Theorem 4.3, we show that if there is a finite place of K which is totally ramified in B, there is never selectivity; that is, every isomorphism class of maximal orders in B admits an embedding of $\mathcal{O}_L$. At the other end of the spectrum, if for each finite place of K, B is split, then the selectivity proportion is either 1 (no selectivity) or $1/[L_0 : K]$. In the case of a central simple algebra B which has partial ramification at some places, the proportion of isomorphism classes which admit an embedding of $\mathcal{O}_L$ will be of the form $m/[L_0 : K]$ for an integer m which is the cardinality of a certain subgroup of $\mathrm{Gal}(L_0/K)$ related to the finite places of K which are partially ramified in B.

Let's begin with the case of a totally ramified prime. This theorem was proven for algebras of odd prime degree in [20], but remains valid for general degree $n \geq 3$.

Theorem 4.3. *Suppose there is a finite place ν of K which is totally ramified in B, that is, $m_\nu = n$. Let $\Omega \subset \mathcal{O}_L$ be any $\mathcal{O}_K$-order. Then every maximal order in B admits an embedding of Ω. In particular, there can never be selectivity.*

Proof. It is enough to show that every maximal order in B admits an embedding of $\mathcal{O}_L$. Since B_ν is a division algebra, there is a unique maximal order $\mathcal{R}_\nu$ in B_ν whose normalizer is all of $B_\nu^\times$ and so $K_\nu^\times$, the norm of the normalizer, is contained in $H_\mathcal{R}$. This means that that ν splits completely in the class field $K(\mathcal{R})$, hence also in $L_0 = K(\mathcal{R}) \cap L$.

On the other hand, by the Albert-Brauer-Hasse-Noether theorem, $m_\nu = n \mid [L_\mathfrak{P} : K_\nu]$ for all places $\mathfrak{P}$ of L lying above ν. This means that ν is inert in L, hence also in L_0. Since L_0/K is unramified (at ν), we have $[L_0 : K] = f(\mathfrak{P}|\nu)$. But ν splits completely in L_0, so $[L_0 : K] = f(\mathfrak{P}|\nu) = 1$, and the result is now immediate from Theorem 3.9. $\square$

To go further, we shall utilize the notion of the distance idele and Proposition 3.8 to characterize those isomorphism classes of maximal orders which admit an embedding of $\mathcal{O}_L$. We have assumed that $\mathcal{O}_L \subset \mathcal{R}$. If there is an embedding of $\mathcal{O}_L$ into a maximal order $\mathcal{E}$, then $\mathcal{O}_L$ is contained in a conjugate maximal order, $\mathcal{E}'$, and by Proposition 3.8, the distance ideles $\delta(\mathcal{R}, \mathcal{E})$ and $\delta(\mathcal{R}, \mathcal{E}')$ are equal. So the idea is to assume that $\mathcal{O}_L$ is contained in maximal orders $\mathcal{R}$ and $\mathcal{E}$, and to consider their distance idele $\delta(\mathcal{R}, \mathcal{E}) \in G_\mathcal{R}$. Recall that $G_\mathcal{R} \cong \mathrm{Gal}(K(\mathcal{R})/K)$, and that we parametrized the isomorphism classes of maximal orders in B with representatives $\mathcal{D}^{\mathbf{a},\mathbf{b},\mathbf{c}}$ having the property that viewing the distance idele as an element of $\mathrm{Gal}(K(\mathcal{R})/K)$ we have (see Proposition 3.4)

$$\delta(\mathcal{R}, \mathcal{D}^{\mathbf{a},\mathbf{b},\mathbf{c}}) = \rho_1^{a_1} \cdots \rho_r^{a_r} \sigma_1^{b_1} \cdots \sigma_s^{b_s} \tau_1^{c_1} \cdots \tau_t^{c_t}.$$

In Theorem 3.9, we see that $\mathcal{O}_L$ is always contained in those representatives where

$$\delta(\mathcal{R}, \mathcal{D}^{\mathbf{a},\mathbf{b},\mathbf{c}}) = \rho_1^{0} \cdots \rho_r^{0} \sigma_1^{b_1} \cdots \sigma_s^{b_s} \tau_1^{c_1} \cdots \tau_t^{c_t},$$

that is, those elements whose distance idele lies in $H = \mathrm{Gal}(K(\mathcal{R})/L_0) \leq G = \mathrm{Gal}(K(\mathcal{R})/K)$. To delve more deeply, we now view $\delta(\mathcal{R}, \mathcal{E})|_{L_0} \in \mathrm{Gal}(L_0/K) \cong G/H$. We sketch the framework we employ.

Recall some notation from the introduction. Given a finite place ν of K, and the local index m_ν, we know that $m_\nu \mid [L_\mathfrak{P} : K_\nu]$ for all places $\mathfrak{P}$ of L lying above ν. Further, we set $r_\mathfrak{P} = [L_\mathfrak{P} : K_\nu]/m_\nu$. Next, we defined:

$$d_\nu = \gcd_{\mathfrak{P}|\nu} r_\mathfrak{P} = \gcd_{\mathfrak{P}|\nu} \frac{[L_\mathfrak{P} : (L_0)_{\mathfrak{P}\cap L_0}][(L_0)_{\mathfrak{P}\cap L_0} : K_\nu]}{m_\nu} \tag{14}$$

$$= \gcd_{\mathfrak{P}|\nu} \frac{[L_\mathfrak{P} : (L_0)_{\mathfrak{P}\cap L_0}] f(\nu; L_0/K)}{m_\nu} \tag{15}$$

$$= \gcd_{\mathfrak{P}|\nu}([L_\mathfrak{P} : (L_0)_{\mathfrak{P}\cap L_0}]) \frac{f(\nu; L_0/K)}{m_\nu}. \tag{16}$$

Now recall that the type distance, $\delta(\mathcal{R}, \mathcal{E})$, is the image of the idele $(\pi_\nu^{td_\nu(\mathcal{R}_\nu, \mathcal{E}_\nu)})_\nu$ in $G_\mathcal{R}$, and viewed as an element of $\mathrm{Gal}(K(\mathcal{R})/K)$ it is a product of (powers of) Artin symbols. So we can view

$$\delta(\mathcal{R},\mathcal{E})|_{L_0} = \prod_{\nu \text{ finite}} (\nu, L_0/K)^{td_\nu(\mathcal{R}_\nu,\mathcal{E}_\nu)},$$

where we recall that the Artin symbol, $(\nu, L_0/K)$, has order equal to the inertia degree $f(\nu; L_0/K)$. Finally, from Theorem 2.1, we know that if $\mathcal{O}_L \subset \mathcal{R}\cap\mathcal{E}$, and ν is unramified in L, then $td_\nu(\mathcal{R}_\nu,\mathcal{E}_\nu)$ will be divisible by d_ν. Now consider Equation (16). If $m_\nu = 1$ (that is, if $B_\nu \cong M_n(K_\nu)$), then d_ν is divisible by $f(\nu; L_0/K)$, the order of $(\nu, L_0/K)$, so that factor in $\delta(\mathcal{R},\mathcal{E})|_{L_0}$ will be trivial. So we see it is here that the partially ramified primes play a critical role in producing a selectivity proportion strictly between $1/[L_0 : K]$ and 1.

Motivated by the above remarks, let $\lambda_1, \ldots, \lambda_\ell$ be the set places which are partially ramified in B.

Remark 4.4. In order to use Theorem 2.1 below, we must also assume that the λ_i are all unramified in L.

For each place, λ_i, we have the quantity d_{λ_i} from Equation (16). Let G_0 be the subgroup of $\mathrm{Gal}(L_0/K)$ generated by the Artin symbols:

$$G_0 = \langle (\lambda_1, L_0/K)^{d_{\lambda_1}}, \ldots, (\lambda_\ell, L_0/K)^{d_{\lambda_\ell}} \rangle \le \mathrm{Gal}(L_0/K).$$

Write f_{λ_i} for $f(\lambda_i; L_0/K)$. From equation (16), we know that

$$d_{\lambda_i} = \gcd_{\mathfrak{P}|\lambda_i}([L_{\mathfrak{P}} : (L_0)_{\mathfrak{P}\cap L_0}])\frac{f_{\lambda_i}}{m_{\lambda_i}},$$

and we know the order of $(\lambda_i, L_0/K)$ is f_{λ_i}. So if $m_{\lambda_i} \mid \gcd_{\mathfrak{P}|\lambda_i}([L_\lambda : (L_0)_{\mathfrak{P}\cap L_0}])$, we know that $(\lambda_i, L_0/K)^{d_{\lambda_i}} = 1 \in G_0$; otherwise it generates a cyclic subgroup of order $f_{\lambda_i}/\gcd(d_{\lambda_i}, f_{\lambda_i})$. For our use below, we want to define maximal orders, $\Gamma^a_{\lambda_i}$, in the local algebra B_{λ_i} with type distance, $td_{\lambda_i}(R_{\lambda_i}, \Gamma^a_{\lambda_i}) = d_{\lambda_i}a$ with $a = 0, 1, \ldots, f_{\lambda_i}/\gcd(d_{\lambda_i}, f_{\lambda_i}) - 1$. We do this in exactly the same way as we did in the previous section just prior to Remark 3.7 where we defined the orders $\mathcal{R}^{a_i}_{\lambda_i}$, so we do not repeat the argument here, although we do reiterate that we are assuming that the places λ_i are unramified in L so as to leverage Theorem 2.1.

Theorem 4.5. *Assume that $\mathcal{O}_L \subset \mathcal{R} \subset B$. For every $\sigma \in G_0$, there exists a maximal order $\mathcal{E}$ in B so that $\mathcal{O}_L \subset \mathcal{E}$, and viewing the distance idele, $\delta(\mathcal{R},\mathcal{E})$, as an element of $\mathrm{Gal}(K(\mathcal{R})/K)$, we have that $\delta(\mathcal{R},\mathcal{E})|_{L_0} = \sigma \in G_0$.*

Proof. Let $\sigma_i = (\lambda_i, L_0/K)^{d_{\lambda_i}} \in G_0$ be a generator of G_0, and write $\sigma = \prod_{i=1}^{\ell} \sigma_i^{a_i}$, where we understand the expression may not be unique. Define a maximal order $\mathcal{E}$ of B via the local-global correspondence by specifying:

$$\mathcal{E}_\nu = \begin{cases} \mathcal{R}_\nu & \text{for } \nu \notin \{\lambda_1, \ldots, \lambda_\ell\}, \\ \Gamma^{a_i}_{\lambda_i} & \text{for } \nu = \lambda_i, \quad i = 1, \ldots, \ell. \end{cases}$$

Then, viewing $\delta(\mathcal{R},\mathcal{E})$ as an element of $\mathrm{Gal}(K(\mathcal{R})/K)$, we have $\delta(\mathcal{R},\mathcal{E}) = \prod_{i=1}^{\ell}(\lambda_i; K(R)/K)^{d_{\lambda_i} a_i}$, so that $\delta(\mathcal{R},\mathcal{E})|_{L_0} = \sigma \in G_0$. $\square$

Remark 4.6. Presuming that $\sigma \neq 1$ in the above theorem, $\mathcal{E} \cong \mathcal{D}^{\mathbf{a},\mathbf{b},\mathbf{c}}$ for some $\mathbf{a} \neq \mathbf{0}$, meaning that the proportion of isomorphism classes admiting an embedding of $\mathcal{O}_L$ is greater than $1/[L_0 : K]$. Indeed, this theorem says that the proportion is at least $|G_0|/[L_0 : K]$.

Now we would like some sort of converse, meaning if there is selectivity, then this is an upper bound as well. We have the following qualified result.

Theorem 4.7. *Assume that $\mathcal{O}_L \subset \mathcal{R} \subset B$. Let $\mathcal{E}$ be another maximal order in B, and let $\delta(\mathcal{R},\mathcal{E})$ denote the distance idele. Assume further, that any place ν for which $td_\nu(\mathcal{R}_\nu,\mathcal{E}_\nu) \not\equiv 0 \pmod{r_\nu}$ is unramified in L. If $\mathcal{O}_L \subset \mathcal{E}$, then $\delta(\mathcal{R},\mathcal{E})|_{L_0} \in G_0$.*

Proof. Let $\delta(\mathcal{R},\mathcal{E}) \in G_{\mathcal{R}} = J_K/H_{\mathcal{R}}$ be the distance idele. Let ν be any place for which $td_\nu(\mathcal{R}_\nu,\mathcal{E}_\nu) \not\equiv 0 \pmod{r_\nu}$. By assumption, we have that ν is unramified in L, and so, by conventions on the type distance, ν is a finite place and not totally ramified in B. Since $O_L \subset \mathcal{E}_\nu$, by Theorem 2.1, we have that $td_\nu(\mathcal{R}_\nu,\mathcal{E}_\nu)$ is divisible by d_ν, which means the local factor of the Artin symbol coming from $\delta(\mathcal{R},\mathcal{E})$ has the form $(\nu; K(\mathcal{R})/K)^{d_\nu \ell}$ for some integer ℓ. So restricted to L_0/K, the Artin symbol becomes $(\nu; L_0/K)^{d_\nu \ell}$. By Equation (16), if ν is unramified in B, then $m_\nu = 1$ which implies $d_\nu \equiv 0 \pmod{f(\nu; L_0/K)}$, but $f(\nu, L_0/K)$ is the order of the Artin symbol $(\nu; L_0/K)$, so this factor is trivial. The only factors left are those which correspond to partially ramified places in B, and so it is clear that $\delta(\mathcal{R},\mathcal{E})|_{L_0} \in G_0$. $\square$

We can summarize the previous two theorems as:

Theorem 4.8. *Let $\lambda_1, \ldots, \lambda_\ell$ be the set of finite places of K which are partially ramified in B. Assume the λ_i are all unramified in L. Let*

$$G_0 = \langle (\lambda_1, L_0/K)^{d_{\lambda_1}}, \ldots, (\lambda_\ell, L_0/K)^{d_{\lambda_\ell}} \rangle \leq \mathrm{Gal}(L_0/K),$$

be the subgroup generated by powers of the Artin symbols $(\lambda_i, L_0/K)$. The proportion of isomorphism classes of maximal orders which admit an embedding of $\mathcal{O}_L$ is at least $\dfrac{|G_0|}{[L_0 : K]}$, and if $L \subseteq K(\mathcal{R})$ (so in particular, L is unramified at all the finite places of K), then the proportion is exactly $\dfrac{|G_0|}{[L_0 : K]}$.

5. An Example

We give a simple example of Theorem 4.8. Computations are done with Magma [8].

Let $K = \mathbb{Q}(\sqrt{-39})$. Then the ideal class group of K is cyclic of order 4, hence the Hilbert class field of K, H_K has Galois group, $\mathrm{Gal}(H_K/K)$, cyclic of order 4. The rational prime 61 splits completely in K, and there are four primes of H_K lying above 61. So put $61\mathcal{O}_K = \mathfrak{p}_1\mathfrak{p}_2$.

Since H_K/K is Galois, the only way for $61\mathcal{O}_{H_K}$ to factor as the product of four distinct primes in H_K is for each of the primes $\mathfrak{p}_i$ to have inertia degrees $f(\mathfrak{p}_1; H_K/K) = f(\mathfrak{p}_2; H_K/K) = 2$.

To construct our central simple algebra, we specify Hasse invariants. Let $m_{\mathfrak{p}_1} = m_{\mathfrak{p}_2} = 2$ and $m_\nu = 1$ for all other places ν of K. Taking Hasse invariants $1/m_\nu$ for all places ν of K, the short exact sequence of Brauer groups (e.g., (32.13) of [25]) guarantees the existence of a degree 4 central simple K-algebra $B = M_2(D)$ having the prescribed Hasse invariants.

Let $L = H_K$. The field L satisfies the conditions of the Albert-Brauer-Hasse-Noether theorem, so L embeds in B as a K-algebra. Now let $\mathcal{R}$ be any maximal order of B which contains $\mathcal{O}_L$, and $K(\mathcal{R})$ the associated class field.

Since K has no real embeddings, its narrow class field and its Hilbert class field coincide, so $K(\mathcal{R}) \subseteq H_K$.

To show the reverse containment, recall that the class field $K(\mathcal{R})$ arises field class field theory via the quotient $J_K/H_{\mathcal{R}}$ where $H_{\mathcal{R}}$ is characterized by information about the local norm of normalizers of the $\mathcal{R}_\nu$ which we characterized in section 3. It is then easy to check that the class group associated to H_K contains $H_{\mathcal{R}}$, so $H_K \subseteq K(\mathcal{R})$.

Thus $L = H_K = K(\mathcal{R}) = L_0$.

We now refer to the notation of Theorem 4.8. We have $\lambda_1 = \mathfrak{p}_1$ and $\lambda_2 = \mathfrak{p}_2$ and via Equation (16), compute $d_{\lambda_1} = d_{\lambda_2} = 1$. So G_0 is generated by the Artin symbols $(\mathfrak{p}_1, H_K/K)$ and $(\mathfrak{p}_2, H_K/K)$ each of which has order 2, but as $\mathrm{Gal}(H_K/K)$ is cyclic of order 4, they must be equal, so that $|G_0| = 2$. So while the standard lower bound for the selectivity proportion is $1/[L_0 : K] = 1/4$, we have $|G_0|/[L_0 : K] = 2/4 = 1/2$.

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