Inverting the Satake map for Sp_n and Applications to Hecke Operators

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Abstract

By compatibly grading the p-part of the Hecke algebra associated to $Sp_n(\mathbb{Z})$ and the subring of $\mathbb{Q}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]$ invariant under the associated Weyl group, we produce a matrix representation of the Satake isomorphism restricted to the corresponding finite dimensional components. In particular, using the elementary divisor theory of integral matrices, we show how to determine the entries of this matrix representation restricted to double cosets of a fixed similitude. The matrix representation is upper-triangular, and can be explicitly inverted.

To address the specific question of characterizing families of Hecke operators whose generating series have "Euler" products, we define $(n+1)$ families of polynomial Hecke operators $t_k^n(p^\ell)$ (in $\mathbb{Q}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]$) for Sp_n whose generating series $\sum t_k^n(p^\ell)v^\ell$ are rational functions of the form $q_k(v)^{-1}$, where q_k is a polynomial in $\mathbb{Q}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}][v]$ of degree $2^k\binom{n}{k}$ (2^n if $k = 0$). For $k = 0$ and $k = 1$ the form of the polynomial is essentially that of the local factors in the spinor and standard zeta functions. For $k > 1$, these appear to be new expressions. Taking advantage of the generating series and our ability to explicitly invert the Satake isomorphism, we explicitly compute the classical operators with the analogous properties in the case of genus 2. It is of interest to note that these operators lie in the full, but not generally the integral, Hecke algebra.

1 Introduction

Hecke theory for modular forms on the symplectic group is still very much in its infancy. There are no doubt many reasons for this, but a comment [7] made to one of the authors struck a chord: The Hecke algebra for Siegel modular forms is a solution looking for a problem. What is meant by this is that the classical Hecke operators were invented to provide a solution to a specific problem: characterizing those modular forms whose Fourier coefficients had multiplicative properties analogous to the divisor and tau functions; in particular

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characterizing those modular forms whose associated L-series had an Euler product of a prescribed type. These Hecke operators did the job admirably leading to a robust theory connecting Fourier coefficients and Hecke eigenvalues, enabling multiplicity-one results and in some sense laying a partial foundation for the Shimura-Taniyama correspondence.

The solution to the above problem (the Hecke operators) was generalized in many ways and to many settings. In the Siegel modular form setting, there are partial generalizations of the Hecke theory. Associated to a simultaneous Hecke eigenform are Satake parameters, which in some sense generalize Hecke eigenvalues, and by which one can associate various L-functions to the eigenform. But the information currently gleaned from the Hecke algebra seems woefully inadequate to produce anything resembling the robust Hecke theory in the elliptic case; that is, our generalized solution does not solve the generalized problem. So we have a solution seeking a problem to solve.

This is not at all to say that some progress has not been made, e.g., using a partial knowledge of Satake parameters to infer complete knowledge [10], or finding correlations between Fourier coefficients and Hecke eigenvalues in degree 2 [4]. Still, we are very far away from a satisfactory general theory, especially one robust enough to produce a theory of newforms for forms associated to congruence subgroups.

This paper takes the step of posing a (Hecke-like) problem and looking for specific Hecke operators which provide a solution to that problem. In particular, we focus on the p-part of the Hecke algebra, and look for families of Hecke operators whose generating series have nice "Euler" product expansions. For Sp_n, we define $(n+1)$ such families, and we find that the solutions are operators which live in the full, but not generally the integral, Hecke algebra. We give recursion relations to define all such operators, and relate them to the standard generators.

The mechanism we employ is to consider the well-known (see e.g., Cartier [3]) Satake isomorphism between the p-part of the Hecke algebra associated to the symplectic group and a polynomial ring invariant under a certain Weyl group. In [1], Andrianov refers to this isomorphism as the spherical map, and give a description of it in terms of right cosets of the double cosets which generate the Hecke algebra.

By first working in the (isomorphic) representation space, we define families of (polynomial) Hecke operators $t_k^n(p^\ell)$, $k = 0, \dots, n$ whose generating series have the form (see Theorem 3.3):

$$\sum_{\ell \geq 0} t_0^n(p^\ell) v^\ell = \left[(1 - x_0 v) \prod_{m=1}^{n} \prod_{1 \leq i_1 < \cdots < i_m \leq n} (1 - x_0 x_{i_1} \cdots x_{i_m} v) \right]^{-1}, \tag{1.1}$$

and for $1 \leq k \leq n$,

$$\sum_{\ell \geq 0} t_k^n(p^\ell) v^\ell = \left[\prod_{1 \leq i_1 < \cdots < i_k \leq n,\ \delta_{i_j} = \pm 1} (1 - x_{i_1}^{\delta_{i_1}} \cdots x_{i_k}^{\delta_{i_k}} v) \right]^{-1}. \tag{1.2}$$

To see the significance of these operators, recall that associated to a simultaneous Hecke eigenfunction F of weight k for $Sp_n(\mathbb{Z})$, are the Satake p-parameters $(\alpha_0, \dots, \alpha_n) = (\alpha_0(p), \dots, \alpha_n(p)) \in \mathbb{C}^{n+1}/W_n$ for each prime p (W_n the associated Weyl group), which generalize the Hecke eigenvalues. The Satake parameters satisfy $\alpha_0(p)^2\alpha_1(p)\cdots\alpha_n(p) = p^{nk-n(n+1)/2}$ and are used to define the spinor and standard zeta functions.

The standard zeta function is defined by $D_F(s) = \prod_p D_{F,p}(p^{-s})^{-1}$ ($\Re(s) > 1$), where

$$D_{F,p}(v) = (1-v)\prod_{m=1}^{n}(1-\alpha_m v)(1-\alpha_m^{-1}v),$$

while the spinor zeta function is defined by $Z_F(s) = \prod_p Z_{F,p}(p^{-s})^{-1}$ ($\Re(s) > nk/2 - n(n+1)/4 + 1$), where

$$Z_{F,p}(v) = (1-\alpha_0 v)\prod_{m=1}^{n}\prod_{1\le i_1<\cdots<i_m\le n}(1-\alpha_0\alpha_{i_1}\cdots\alpha_{i_m}v).$$

For $k = 0$, the expression (1.1) clearly corresponds to the local factor of the spinor zeta function. When $k = 1$, the expression (1.2) is simply $\sum_{\ell\ge 0} t_1^n(p^\ell)v^\ell = \left[\prod_{m=1}^{n}(1-x_m v)(1-x_m^{-1}v)\right]^{-1}$ which (up to an initial "zeta" factor) corresponds to the local factor of the standard zeta function. Except for $k=0$ and $k=1$, the Hecke operators, $t_k^n(p^\ell)$, give rise to new "zeta" functions which may also be of interest in the context of Siegel modular forms.

After defining operators of interest in the representation space, we put a natural and compatible grading on both the local Hecke algebra and on the ring of symmetric polynomials, and show that the Satake isomorphism restricts to one between corresponding finite-dimensional components. Using the elementary divisor theory of integral matrices, we show how to determine the entries of this matrix representation restricted to double cosets of a fixed similitude. The matrix representation is upper-triangular, and can be explicitly inverted.

By using an explicit inverse in the case of genus 2, we pull back the Hecke operators in the polynomial ring to define classical Hecke operators in terms of double cosets whose generating series have the same product representation. We indicate the recursion relations required to define all the operators.

An interesting feature is that unlike the elliptic case, not all these operators actually lie in the integral Hecke algebra. That is, these results suggest that the operators which may eventually lead to a robust Hecke theory in the Siegel setting may not be direct analogs of those in the elliptic case, and we may need to recast the elliptic case in a new light to see a natural generalization.

2 The Satake Isomorphism

The Satake isomorphism is a map between a local Hecke algebra and a ring of symmetric polynomials. In this section we define the appropriate Hecke algebra, describe the symmetry group corresponding to Sp_n, and give a few properties of the Satake map.

2.1 Hecke Algebras and Polynomial Rings

To set the notation, we begin with the global Hecke algebra over $\mathbb{Q}$; most of this is standard (see e.g., Chapter 3 of [1]). Let I denote the $n \times n$ identity matrix and J the $2n \times 2n$ matrix $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. Let $S = GSp_n^+(\mathbb{Q}) \subset GL_{2n}(\mathbb{Q})$ be the group of of symplectic similitudes with scalar factor $r(\alpha) \in \mathbb{Q}_+^\times$: defined by $GSp_n^+(\mathbb{Q}) = \{\alpha \in M_{2n}(\mathbb{Q}) \mid {}^t\alpha J \alpha = r(\alpha) J\}$, where ${}^t\alpha$ denotes the transpose of α. The number $r(\alpha)$ will be called the similitude of α. Denote by $\Gamma = \Gamma_n = Sp_n(\mathbb{Z}) \subset SL_{2n}(\mathbb{Z})$ those elements of $GSp_n^+(\mathbb{Q})$ having similitude 1.

For computational purposes it is often convenient to realize

$$\begin{aligned} GSp_n^+(\mathbb{Q}) &= \{\alpha = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in M_{2n}(\mathbb{Q}) \mid A^tC = C^tA, B^tD = D^tB, A^tD - C^tB = r(\alpha)I_{2n}\} \\ &= \{\alpha = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in M_{2n}(\mathbb{Q}) \mid AB^t = B^tA, CD^t = DC^t, AD^t - BC^t = r(\alpha)I_{2n}\}. \end{aligned}$$

Let $H = H(\Gamma, S)$ denote the rational Hecke algebra associated to the pair Γ and S. As a vector space H is generated by all double cosets $\Gamma\xi\Gamma$ ($\xi \in S$), and we turn H into an algebra by defining the multiplication law as follows: Given $\xi_1, \xi_2 \in S$, define

$$\Gamma\xi_1\Gamma \cdot \Gamma\xi_2\Gamma = \sum_{\xi} c(\xi)\Gamma\xi\Gamma, \tag{2.1}$$

where the sum is over all double cosets $\Gamma\xi\Gamma \subseteq \Gamma\xi_1\Gamma\xi_2\Gamma$, and the $c(\xi)$ are nonnegative integers (see [9]). There is an alternate characterization of the Hecke algebra which will be convenient as well. Let $L(\Gamma, S)$ be the rational vector space with basis consisting of right cosets $\Gamma\xi$ for $\xi \in S$. The Hecke algebra can be thought of as those elements of $L(\Gamma, S)$ which are right invariant under the action of Γ. Thus we can and will think of a double coset as the disjoint union of right cosets $\Gamma\xi\Gamma = \cup\Gamma\xi_\nu$ and as the sum of the same cosets $\sum \Gamma\xi_\nu \in L(\Gamma, S)$.

The global Hecke algebra, H, is generated by local Hecke algebras, H_p, one for each prime p, obtained as above by replacing S by $S_p = S \cap GL_{2n}(\mathbb{Z}[p^{-1}])$ in the above construction. H_p is generated by double cosets $\Gamma\xi\Gamma$ with ξ of the form $\operatorname{diag}(p^{a_1}, \dots, p^{a_n}; p^{b_1}, \dots, p^{b_n})$ where $a_1 \le \cdots \le a_n \le b_n \le \cdots \le b_1$ are integers with $p^{a_i+b_i} = r(\xi)$ for all i. It is often useful to consider the "integral" Hecke algebra $\underline{H}_p$ generated by all ξ as above with $\xi = \operatorname{diag}(p^{a_1}, \dots, p^{a_n}; p^{b_1}, \dots, p^{b_n}) \in M_{2n}(\mathbb{Z})$.

The integral Hecke algebra $\underline{H}_p$ is generated by the $(n+1)$ Hecke operators

$$T(p) = \Gamma \begin{pmatrix} I_n & 0 \\ 0 & pI_n \end{pmatrix} \Gamma$$

and for $k = 1, \ldots, n$,

$$T_k^n(p^2) = T_k(p^2) = \Gamma \begin{pmatrix} I_{n-k} & 0 & 0 & 0 \\ 0 & pI_k & 0 & 0 \\ 0 & 0 & p^2 I_{n-k} & 0 \\ 0 & 0 & 0 & pI_k \end{pmatrix} \Gamma,$$

while the Hecke algebra H_p is generated by the $(n+1)$ elements above together with the element $T_n(p^2)^{-1} = (pI_{2n})^{-1}$. We also identify $T_0^n(p^2) = \Gamma\,\mathrm{diag}(1, \ldots, 1; p^2, \ldots, p^2)\Gamma$.

Let $\mathfrak{S}_n$ denote the symmetric group on n letters, and W_n be the group of $\mathbb{Q}$-automorphisms of the rational function field $\mathbb{Q}(x_0, \ldots, x_n)$ generated by all permutations of the variables $x_1, \ldots, x_n$ and by the automorphisms $\tau_1, \ldots, \tau_n$ which are given by:

$$\tau_i(x_0) = x_0 x_i, \qquad \tau_i(x_i) = x_i^{-1}, \qquad \tau_i(x_j) = x_j \quad (0 < j \neq i).$$

W_n is a signed permutation group, in particular, $W_n = \langle \tau_i \rangle \rtimes \mathfrak{S}_n \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n \cong C_n$ where C_n is Coxeter group associated to the spherical building for $Sp_n(\mathbb{Q}_p)$. Finally let $\mathbb{Q}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]^{W_n}$ (and analogous polynomial rings), be the set of polynomials in the variables $x_i^{\pm 1}$ invariant under the action of the group W_n. It is worth noting for our future use (see [1]) that $\mathbb{Q}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]^{W_n} \cong \mathbb{Q}[x_0, \ldots, x_n]^{W_n}[(x_0^2 x_1 \cdots x_n)^{-1}]$.

The Satake isomorphism (see [1]) establishes the isomorphism $H_p \cong \mathbb{Q}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]^{W_n}$ in such a way that the integral (local) Hecke algebra $\underline{H}_p$ is isomorphic to the integral polynomial ring $\mathbb{Q}[x_0, \ldots, x_n]^{W_n}$.

2.2 Tools for analyzing the Satake Isomorphism

In [1], Andrianov refers to the Satake isomorphism as the spherical map $\Omega : \underline{H}_p \to \mathbb{Q}[x_0, \ldots, x_n]^{W_n}$ and give the following characterization. Let $\Gamma\xi\Gamma \in \underline{H}_p$. Decompose $\Gamma\xi\Gamma$ into right cosets $\cup_{i=1}^{\nu} \Gamma\alpha_i$ and define the image of the double coset to be the sum of the images on right cosets:

$$\Omega(\Gamma\xi\Gamma) = \Omega\left(\cup_{i=1}^{\nu} \Gamma\alpha_i\right) = \sum_{i=1}^{\nu} \Omega(\Gamma\alpha_i).$$

By Lemma 3.2.7 of [1], any right coset in such a decomposition has as unique representative in the form

$$\Gamma\alpha = \Gamma \begin{pmatrix} p^r({}^tM)^{-1} & N \\ 0 & M \end{pmatrix} \text{ where } M = \begin{pmatrix} p^{a_1} & * & \cdots & * \\ 0 & p^{a_2} & \cdots & * \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \cdots & p^{a_n} \end{pmatrix},$$

and they define

$$\Omega(\Gamma\alpha) = x_0^r \prod_{i=1}^{n} (x_i p^{-i})^{a_i} = p^{-\sum i a_i} x_0^r x_1^{a_1} \cdots x_n^{a_n}.$$

Deriving a matrix representation of the Satake map is achieved through a series of steps. To understand the image of Ω, we must relate the symplectic elementary divisors of the double coset $\Gamma\xi\Gamma$ to the GL_n elementary divisors of the lower right blocks (M) which make up its right coset representatives. Even more precisely than stated above, we characterize the structure of the right coset representatives appearing in the decomposition of a given double coset. We then introduce a natural and compatible grading on the infinite dimensional Hecke and polynomial algebras, and derive a matrix representation on corresponding components which is upper triangular. We conclude by computing some examples.

2.3 Canonical Forms and Matrix Equivalence

Let $G = G_n = GL_n(\mathbb{Z})$ and $K = K_n = GL_n(\mathbb{Q})$. For $A, B \in K$ we write $A \sim_G B$ to mean there are $U, V \in G$ so that $UAV = B$. For rational numbers r, s, write $r \mid s$ if $s = rm$ for some $m \in \mathbb{Z}$. It is well known from standard elementary divisor theory that we have the following Smith normal form for matrices in K.

Proposition 2.1 (Lemma 3.2.2 in [1]). *Let $A \in K$. Then there exists a unique matrix* $\operatorname{diag}(e_1, \dots, e_n)$ *so that $e_i \in \mathbb{Q}_+$, $e_{i+1} \mid e_i$ and*

$$A \sim_G \operatorname{diag}(e_1, \dots, e_n).$$

The matrix described in Proposition 2.1 is called the *elementary divisor form of A* and will be denoted $\operatorname{ed}(A)$. The proof of this result can be found in [1] among other places; note, however, that the order we give the e_i is opposite to that in [1].

Recall that $S = GSp_n^+(\mathbb{Q})$ and $\Gamma = Sp_n(\mathbb{Z})$. For $\alpha, \beta \in S$ we write $\alpha \sim_\Gamma \beta$ to mean there are matrices $\gamma, \delta \in \Gamma$ so that $\gamma\alpha\delta = \beta$. Analogous to the elementary divisor form of a matrix in K, we have the symplectic divisor form $\operatorname{sd}(\alpha)$ of a matrix in S (see [1]):

Proposition 2.2 (Lemma 3.3.6 in [1]). *Let $\alpha \in S \cap M_{2n}(\mathbb{Z})$ have similitude $r(\alpha)$. Then there exists a unique matrix* $\operatorname{diag}(d_1, \dots, d_n; e_1, \dots, e_n)$ *with $d_i, e_i \in \mathbb{Z}_+$, satisfying $d_i \mid d_{i+1}, d_n \mid e_n, e_{i+1} \mid e_i$, $d_i e_i = r(\alpha)$, and*

$$\alpha \sim_\Gamma sd(\alpha) = \operatorname{diag}(d_1, \dots, d_n; e_1, \dots, e_n).$$

Let $\mathcal{T}_{k,n}$ be the set of all k-tuples $\mathbf{i} = (i_1, \dots, i_k)$ with $1 \le i_1 < i_2 < \cdots < i_k \le n$. For $\mathbf{t}, \mathbf{s} \in \mathcal{T}_{k,n}$ let $A(\mathbf{t}, \mathbf{s})$ be the minor of the matrix whose rows are determined by the entries of $\mathbf{t}$ and whose columns by $\mathbf{s}$. The following computational tool introduced in [6] is central to what follows.

Definition 2.3. *Let $A \in K$ and $1 \le k \le n$. The kth determinantal divisor of A, denoted $d_k(A)$, is the greatest common divisor of the $\binom{n}{k}^2$ numbers $A(\mathbf{t}, \mathbf{s})$ for $\mathbf{t}, \mathbf{s} \in \mathcal{T}_{k,n}$. If $A(\mathbf{t}, \mathbf{s}) = 0$ for all $\mathbf{t}, \mathbf{s} \in \mathcal{T}_{k,n}$, then $d_k(A) = 0$.*

Proposition 2.4. *Let $\alpha, \beta \in S$. Then the following are equivalent:*

i) $\alpha \sim_\Gamma \beta$,
ii) $d_k(\alpha) = d_k(\beta)$ *for all* $1 \le k \le 2n$,
iii) $ed(\alpha) = ed(\beta)$,
iv) $sd(\alpha) = sd(\beta)$.

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are established in [6].

(iii) implies (iv): We show that $\mathrm{sd}(\alpha)$ (respectively $\mathrm{sd}(\beta)$) is the same matrix as $\mathrm{ed}(\alpha)$ (respectively $\mathrm{ed}(\beta)$) up to the same permutation of the entries. Suppose $\mathrm{ed}(\alpha) = \mathrm{ed}(\beta) = \mathrm{diag}(f_1, \dots, f_n, f_{n+1}, \dots, f_{2n})$ where $f_{i+1} \mid f_i$. Also, with the notation of Proposition 2.2, let $\mathrm{sd}(\alpha) = \mathrm{diag}(d_1, \dots, d_n; e_1, \dots, e_n)$ (respectively, $\mathrm{sd}(\beta) = \mathrm{diag}(d'_1, \dots, d'_n; e'_1, \dots, e'_n)$) where $e_{i+1} \mid e_i, d_n \mid e_n$ and $d_i \mid d_{i+1}$ (respectively, $e'_{i+1} \mid e'_i, d'_n \mid e'_n$ and $d'_i \mid d'_{i+1}$). By rearranging and renaming the entries of $\mathrm{sd}(\alpha)$ and $\mathrm{sd}(\beta)$ we get

$$\widetilde{\mathrm{sd}}(\alpha) = \mathrm{diag}(e_1, \dots, e_n, d_n, \dots, d_1) = (c_1, \dots, c_{2n})$$

and

$$\widetilde{\mathrm{sd}}(\beta) = \mathrm{diag}(e'_1, \dots, e'_n, d'_n, \dots, d'_1) = (c'_1, \dots, c'_{2n}).$$

Note that both $c_{i+1} \mid c_i$ and $c'_{i+1} \mid c'_i$.

Since $\Gamma = Sp_n(\mathbb{Z}) \subset G_{2n} = GL_{2n}(\mathbb{Z})$ we know that $\Gamma\alpha\Gamma \subset G_{2n}\alpha G_{2n}$. Since $\mathrm{sd}(\alpha) \in \Gamma\alpha\Gamma$ and $\widetilde{\mathrm{sd}}(\alpha)$ is merely the product of $\mathrm{sd}(\alpha)$ with permutation matrices in G_{2n} we see that $\widetilde{\mathrm{sd}}(\alpha) \in G_{2n}\alpha G_{2n}$. On the other hand, because $\mathrm{ed}(\alpha) = \mathrm{ed}(\beta)$ we know that $\mathrm{ed}(\beta) \in G_{2n}\alpha G_{2n}$ and in particular, $\beta \in G_{2n}\alpha G_{2n}$. This last statement implies $\mathrm{sd}(\beta) \in G_{2n}\alpha G_{2n}$ and by a similar argument as above, $\widetilde{\mathrm{sd}}(\beta) \in G_{2n}\alpha G_{2n}$.

Since $\mathrm{ed}(\alpha)$ is the only matrix of the form prescribed by Proposition 2.1 and both $\widetilde{\mathrm{sd}}(\alpha)$ and $\widetilde{\mathrm{sd}}(\beta)$ have the same form, we conclude

$$\mathrm{ed}(\alpha) = \widetilde{\mathrm{sd}}(\alpha) = \widetilde{\mathrm{sd}}(\beta).$$

By the way $\widetilde{\mathrm{sd}}(\alpha)$ and $\widetilde{\mathrm{sd}}(\beta)$ were constructed, we conclude $\mathrm{sd}(\alpha) = \mathrm{sd}(\beta)$.

(iv) implies (i): By Proposition 2.2 we know $\alpha \sim_\Gamma \mathrm{sd}(\alpha)$ and $\beta \sim_\Gamma \mathrm{sd}(\beta)$. Since $\mathrm{sd}(\alpha) = \mathrm{sd}(\beta)$ we have $\alpha \sim_\Gamma \beta$. □

2.4 Grading the Hecke and Polynomial Algebras

We first establish a natural grading of the local integral Hecke algebra. Recall $S_p = GSp_n^+(\mathbb{Q}) \cap GL_{2n}(\mathbb{Z}[p^{-1}])$, and let $S_p(\ell) = \{\alpha \in S_p \mid r(\alpha) = p^\ell, \ell \ge 0\}$. We call an element $\alpha \in S_p(\ell)$ an *integral similitude of degree* ℓ.

Definition 2.5. *Denote by* $\underline{H}_p(\Gamma, S_p(\ell))$ *the vector subspace of* $\underline{H}_p = \underline{H}_p(\Gamma, S_p)$ *generated by* $\{\Gamma\alpha\Gamma : \alpha \in S_p(\ell)\}$.

It is obvious that $\underline{H}_p(\Gamma, S_p(\ell))$ is finite dimensional since it is spanned by matrices of similitude p^ℓ in symplectic-divisor form (Proposition 2.2); an explicit basis is given in Proposition 2.20. It is immediate from the definition that the structure of the local Hecke algebra is given by the following proposition.

Proposition 2.6. $\underline{H}_p = \bigoplus_{\ell \geq 0} \underline{H}_p(\Gamma, S_p(\ell))$, *where the sum is over all non-negative integers* ℓ.

Recall that $\mathfrak{S}_n$ denotes the symmetric group on n letters and viewing this group as a set of automorphisms of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$, we let $\mathbb{Q}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$ denote the standard ring of symmetric (i.e. $\mathfrak{S}_n$-invariant) polynomials. Analogously, we let $\mathbb{Q}[x_0, \ldots, x_n]^{W_n}$ denote the ring of polynomials in $(n+1)$ variables invariant under the action of the group of automorphisms in W_n (see Section 2.1). We wish to introduce a grading on the polynomial ring $\mathbb{Q}[x_0, \ldots, x_n]^{W_n}$. We call a polynomial $f \in \mathbb{Q}[x_0, \ldots, x_n]^{W_n}$ a *W_n-invariant polynomial of similitude r* if it can be written as $f = x_0^r g$ for some $g \in \mathbb{Q}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$

Definition 2.7. *Let $r \geq 0$ be an integer, and let $\mathbb{Q}_r[x_0, \ldots, x_n]^{W_n}$ denote the subspace of $\mathbb{Q}[x_0, \ldots, x_n]^{W_n}$ spanned by all W_n-invariant polynomials of similitude r.*

Analogous to what we have done with the Hecke algebra, our goal is to show that $\mathbb{Q}[x_0, \ldots, x_n]^{W_n}$ is the direct sum of the $\mathbb{Q}_r[x_0, \ldots, x_n]^{W_n}$. We introduce some notation here which will be used below and in subsequent sections.

Denote elements of $\mathbb{Z}^n$ by $\mathbf{b} = (b_1, \ldots, b_n)$; in particular, put $\mathbf{1} = (1, \ldots, 1)$, and let $\leq$ denote the lexicographic order on $\mathbb{Z}^n$. For $\mathbf{b} \in \mathbb{Z}^n$ we write $p^{\mathbf{b}}$ to mean the matrix $\operatorname{diag}(p^{b_1}, \ldots, p^{b_n})$ and $x^{\mathbf{b}}$ the monomial $x_1^{b_1} \cdots x_n^{b_n}$. Define the set

$$\mathcal{C}_n(r) = \left\{ \mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{Z}^n : r \geq b_1 \geq \cdots \geq b_n \geq 0 \right\}.$$

We begin generally by characterizing the action of W_n on $\mathbb{Q}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]$:

Proposition 2.8. *Let $\mathbf{a} \in \mathcal{C}_n(r)$ for $r \geq 0$. Under the action of W_n we obtain the following orbit:*

$$\operatorname{Orbit}_{W_n}(x_0^r x^{\mathbf{a}}) = \left\{ x_0^r x_{\sigma(1)}^{\epsilon_{\sigma(1)}} \cdots x_{\sigma(n)}^{\epsilon_{\sigma(n)}} : \sigma \in \mathfrak{S}_n, \epsilon_i = a_i \text{ or } r - a_i \right\}.$$

In particular, W_n sends a polynomial with factor x_0^r to another with factor x_0^r.

Proof. We see the equality directly: The elements of $\mathfrak{S}_n$ leave x_0 unchanged, and applying τ_i to $x_0^r x^{\mathbf{a}}$ we get

$$\begin{aligned} \tau_i(x_0^r x^{\mathbf{a}}) &= x_0^r x_i^r x_1^{a_1} \cdots x_i^{-a_i} \cdots x_n^{a_n} \\ &= x_0^r x_1^{a_1} \cdots x_i^{r-a_i} \cdots x_n^{a_n}. \end{aligned}$$

Since W_n is generated by the τ_i and $\sigma \in \mathfrak{S}_n$ we deduce the desired equality. ∎

Now we can state:

Proposition 2.9. *The algebra of W_n-invariant polynomials is the direct sum over nonnegative r of the W_n-invariant polynomials of similitude r; i.e.,*

$$\mathbb{Q}[x_0, \ldots, x_n]^{W_n} = \oplus_{r \geq 0} \mathbb{Q}_r[x_0, \ldots, x_n]^{W_n}.$$

Proof. Suppose that $f \in \mathbb{Q}[x_0, \ldots, x_n]^{W_n}$. We parse the set of monomials of f according to the highest power of x_0^r that appears in each monomial. If $0 \leq i_1 < i_2 < \cdots < i_k$ are the various powers of x_0 that appear in the monomials of f then

$$f = \sum_{j=1}^{k} x_0^{i_j} f_j, \ f_j \in \mathbb{Q}[x_1, \ldots, x_n].$$

Let σ be a permutation of the subscripts of $x_1, \ldots, x_n$ and f^σ denote the action on f by σ. Then, since, $f^\sigma = f$ we see that $(x_0^{i_j} f_j)^\sigma = x_0^{i_j} f_j$ on the one hand and on the other we see that $(x_0^{i_j} f_j)^\sigma = x_0^{i_j} f_j^\sigma$. Thus $f_j^\sigma = f_j$ and $f_j \in \mathbb{Q}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$. From this we conclude $f \in \sum_{r \geq 0} \mathbb{Q}_r[x_0, \ldots, x_n]^{W_n}$.

By standard linear algebra, we know that monomials in $\mathbb{Q}[x_0, \ldots, x_n]^{W_n}$ with different powers of x_0 are linearly independent. Since we are grading $\mathbb{Q}[x_0, \ldots, x_n]^{W_n}$ according to these powers, the theorem is proved. □

2.5 Double Cosets of Similitude r

We say a matrix $M = (m_{ij}) \in GL_n(\mathbb{Q})$ is *reduced* if it is upper triangular and $0 \leq m_{ij} < m_{jj}$ for $1 \leq i < j \leq n$; denote $({}^tM)^{-1}$ by M^*.

The goal of this section is to give a refinement of Lemma 3.3.11 in [1], characterizing a set of right coset representatives for a given double coset:

Proposition 2.10. *For $r \geq 0$, let $\xi \in S_p(r) \cap M_{2n}(\mathbb{Z})$; then there exists a $\mathbf{b} \in \mathcal{C}_n(r)$ for which*

$$\Gamma \xi \Gamma = \Gamma \begin{pmatrix} p^{r\mathbf{1}-\mathbf{b}} & 0 \\ 0 & p^{\mathbf{b}} \end{pmatrix} \Gamma.$$

Furthermore, $\Gamma \xi \Gamma$ has a decomposition into right cosets of the form

$$\Gamma \begin{pmatrix} p^r M^* & N \\ 0 & M \end{pmatrix}$$

where

1. *$M \in M_n(\mathbb{Z})$ is of the form*

$$\begin{pmatrix} p^{a_1} & * & \cdots & * \\ 0 & p^{a_2} & \cdots & * \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \cdots & p^{a_n} \end{pmatrix}$$

   and is reduced,

2. $p^r M^* \in M_n(\mathbb{Z})$ is of the form

$$\begin{pmatrix} p^{r-a_1} & 0 & \cdots & 0 \\ * & p^{r-a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ * & * & \cdots & p^{r-a_n} \end{pmatrix}$$

3. the possible matrices N in each right coset are completely determined by the matrices M in the decomposition,

4. for some $\sigma \in \mathfrak{S}_n$, $\mathbf{a} = (a_{\sigma(1)}, \dots, a_{\sigma(n)}) \in \mathcal{C}_n(r)$ is such that $\mathbf{a} \leq \mathbf{b}$ in the lexicographic order on $\mathcal{C}_n(r)$.

Proof. Let $\Delta = S_p(r) \cap M_{2n}(\mathbb{Z})$. By Proposition 2.2 we know that each matrix $\xi \in \Delta$ has a symplectic divisor form $\mathrm{sd}(\xi)$. Moreover, since $\xi \in \Delta$, evidently $\mathrm{sd}(\xi) \in \Delta$ and, in particular, is integral and has similitude p^r. Suppose $\mathrm{sd}(\xi) = \mathrm{diag}(p^{d_1}, \dots, p^{d_n}; p^{b_1}, \dots, p^{b_n})$. Then, since $\mathrm{sd}(\xi)$ is integral $b_i, d_i \geq 0$. Furthermore, since $\mathrm{sd}(\xi)$ has similitude p^r, we know $d_i = r - b_i$. Finally, by the definition of $\mathrm{sd}(\xi)$ we know that $b_1 \geq \cdots \geq b_n \geq r - b_n \geq \cdots \geq r - b_1 \geq 0$ and hence $\mathbf{b} = (b_1, \dots, b_n) \in \mathcal{C}_n(r)$.

Suppose $\Gamma\alpha$ is a right coset that appears in the decomposition of the double coset $\Gamma\mathrm{sd}(\xi)\Gamma$. By Lemma 3.3.11 in [1], we know that there exists $\gamma \in \Gamma$ so that $\gamma\alpha = \left(\begin{smallmatrix} p^r M^* & N \\ 0 & M \end{smallmatrix}\right)$ for M and $p^r M^*$ as described in the statement of the theorem. This proves the first two assertions of the theorem. The third statement follows from Lemma 3.3.33 in [1].

Before we prove the final statement, note that by the way we constructed $\mathbf{b}$ above, we have $b_1 \geq \cdots \geq b_n \geq r - b_n \geq \cdots \geq r - b_1 \geq 0$ and that $\mathbf{d} = (r - b_n, \dots, r - b_1) \in \mathcal{C}_n(r)$. Thus $\mathbf{d}$ is minimal in the following sense: if $\mathbf{c} = (c_1, \dots, c_n) \in \mathcal{C}_n(r)$ so that for each i, $c_i = b_j$ or $c_i = r - b_j$ for some j, then $\mathbf{d} \leq \mathbf{c}$.

Note that since ξ is integral, $\Gamma\xi\Gamma = \Gamma\mathrm{sd}(\xi)\Gamma = \cup_{i=1}^{\nu} \Gamma\alpha_i$ where all the α_i are integral matrices as well. Let $\Gamma\alpha$ be a right coset in the decomposition of $\Gamma\xi\Gamma$; then we may assume $\alpha = \left(\begin{smallmatrix} p^r M^* & N \\ 0 & M \end{smallmatrix}\right)$ where $M = \begin{pmatrix} p^{a_1} & * & \cdots & * \\ 0 & p^{a_2} & \cdots & * \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \cdots & p^{a_n} \end{pmatrix}$ is reduced. Since α is integral, so are M and $p^r M^*$ (in particular p^{a_i} and p^{r-a_i} are integers). Thus $0 \leq a_i \leq r$. Choose $\sigma \in \mathfrak{S}_n$ so that $a_{\sigma(1)} \geq \cdots \geq a_{\sigma(n)} \geq 0$. Thus $\mathbf{a} := (a_{\sigma(1)}, \dots, a_{\sigma(n)}) \in \mathcal{C}_n(r)$. All that remains to be shown is that $\mathbf{a} \leq \mathbf{b}$.

We know that

$$\alpha = \begin{pmatrix} p^r M^* & N \\ 0 & M \end{pmatrix} \sim_\Gamma \begin{pmatrix} p^{r\mathbf{1}-\mathbf{b}} & 0 \\ 0 & p^{\mathbf{b}} \end{pmatrix} = \mathrm{sd}(\xi)$$

and, in particular, by Theorem 2.4 they have the same determinantal divisors.

For each $1 \leq k \leq n$, the kth determinantal divisor of $\mathrm{sd}(\xi)$ is given by

$$d_k(\mathrm{sd}(\xi)) = p^{\sum_1^k r - b_i}$$

since $\mathbf{d} = (r - b_n, \dots, r - b_1)$ is minimal in the sense described above. We also know

$$d_k(\alpha) = d_k(\mathrm{sd}(\xi)) \text{ which divides } p^{\sum_1^k r - a_{\sigma(i)}}$$

since $p^{\sum_1^k r-a_{\sigma(i)}}$ is the determinant of a particular $k\times k$ submatrix of α. From this we conclude that for each $1\le k\le n$, $\displaystyle\sum_{i=1}^k r-b_i \le \sum_{i=1}^k r-a_{\sigma(i)}$, or equivalently, $\displaystyle\sum_{i=1}^k a_{\sigma(i)} \le \sum_{i=1}^k b_i$.

In particular, $a_{\sigma(1)} \le b_1$. If $a_{\sigma(1)} < b_1$ we immediately conclude $\mathbf{a} \le \mathbf{b}$; otherwise $a_{\sigma(1)} = b_1$. By equation (2.5) we know $a_{\sigma(1)} + a_{\sigma(2)} \le b_1 + b_2 = a_{\sigma(1)} + b_2$ and thus $a_{\sigma(2)} \le b_2$. Continuing in this way we see that $\mathbf{a} \le \mathbf{b}$. □

The following corollary explicitly connects the elementary divisor theory of $GL_n(\mathbb{Z})$ as used in [8] with the symplectic divisor theory we are developing here:

Corollary 2.11. *Suppose for* $\mathbf{b}\in\mathcal{C}_n(r)$ *and for*

$$M = \begin{pmatrix} p^{a_1} & * & \cdots & * \\ 0 & p^{a_2} & \cdots & * \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \cdots & p^{a_n} \end{pmatrix} \in M_n(\mathbb{Z}),$$

the coset $\Gamma\begin{pmatrix} p^rM^* & N \\ 0 & M\end{pmatrix}$ *appears in the decomposition of the double coset* $\Gamma\begin{pmatrix} p^{r\mathbf{1}-\mathbf{b}} & 0 \\ 0 & p^{\mathbf{b}}\end{pmatrix}\Gamma$. *If we write* $p^{\mathbf{e}} = \mathrm{diag}(p^{e_1},\ldots,p^{e_n})$ *to denote the elementary divisor form of* M, *then* $\mathbf{e}\in\mathcal{C}_n(r)$ *and there exists a* $\sigma\in\mathfrak{S}_n$ *for which* $(a_{\sigma(1)},\ldots,a_{\sigma(n)}) := \mathbf{a}\le\mathbf{e}\le\mathbf{b}$.

Proof. The proof of the corollary follows the proof of Proposition 2.10. □

2.6 The Matrix Representation of the Satake map, Ω

In this section, we give natural bases for $\underline{H}_p(\Gamma, S_p(r))$ and $\mathbb{Q}_r[x_0,\ldots,x_n]^{W_n}$ and explicitly compute the matrix of Ω with respect to these bases. We begin by showing how to write the image of a double coset under Ω as a power of x_0 times the sum of $\mathfrak{S}_n$-invariant polynomials

Recall that the Satake map $\Omega : \underline{H}_p \to \mathbb{Q}[x_0,\ldots,x_n]^{W_n}$ is defined on right cosets [1]: if

$$\Gamma\alpha = \Gamma\begin{pmatrix} p^rM^* & N \\ 0 & M\end{pmatrix} \text{ where } M = \begin{pmatrix} p^{a_1} & * & \cdots & * \\ 0 & p^{a_2} & \cdots & * \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \cdots & p^{a_n} \end{pmatrix},$$

then

$$\Omega(\Gamma\alpha) = x_0^r\prod_{i=1}^n (x_ip^{-i})^{a_i} = p^{-\sum ia_i}x_0^rx_1^{a_1}\cdots x_n^{a_n}.$$

To find the image of a given double coset $\Gamma\xi\Gamma$, we decompose $\Gamma\xi\Gamma$ into right cosets $\Gamma\alpha_i$ and define the image of the union to be the sum of the images: $\Omega(\Gamma\xi\Gamma) = \Omega\left(\cup_{i=1}^{\nu}\Gamma\alpha_i\right) = \sum_{i=1}^{\nu}\Omega\left(\Gamma\alpha_i\right)$.

Notation 2.12. *To make the bookkeeping a bit easier, we introduce the following notation. For* $r\in\mathbb{Z}, r\ge 0$, *and* $\mathbf{a},\mathbf{b},\mathbf{e}\in\mathcal{C}_n(r)$,

- *Let* $\mathcal{M}(\mathbf{a},\mathbf{e},r)$ *be the set of matrices* $M \in M_n(\mathbb{Z})$ *where* M *is reduced,* $p^r M^* \in M_n(\mathbb{Z})$*,* $ed(M) = p^{\mathbf{e}}$ *and the diagonal entries of* M *are* $p^{a_1}, \ldots, p^{a_n}$*.*

  Also let, $\mathcal{M} = \mathcal{M}_{\mathbf{b}}(\mathbf{a},\mathbf{e},r)$ *be the subset of* $\mathcal{M}(\mathbf{a},\mathbf{e},r)$ *for which there exists an* $N \in M_n(\mathbb{Z})$ *so that the right coset* $\Gamma \begin{pmatrix} p^r M^* & N \\ 0 & M \end{pmatrix}$ *appears in the decomposition of* $\Gamma \begin{pmatrix} p^{r\mathbf{1}-\mathbf{b}} & 0 \\ 0 & p^{\mathbf{b}} \end{pmatrix} \Gamma$*. We denote the cardinality of* $\mathcal{M}$ *by* $\#\mathcal{M} = \#\mathcal{M}_{\mathbf{b}}(\mathbf{a},\mathbf{e},r)$*.*
- *For* $M \in \mathcal{M}$ *we denote by* $\mathcal{N}(M)$ *the set of* $N \in M_n(\mathbb{Z})$ *for which* ${}^tNM$ *is symmetric. Also, by* $\mathcal{N} = \mathcal{N}_{\mathbf{b}}(M)$ *we denote the set of* $N \in \mathcal{N}(M)$ *for which* $\Gamma \begin{pmatrix} p^r M^* & N \\ 0 & M \end{pmatrix}$ *appears in the decomposition of the double coset* $\Gamma \begin{pmatrix} p^{r\mathbf{1}-\mathbf{b}} & 0 \\ 0 & p^{\mathbf{b}} \end{pmatrix} \Gamma$*. We denote the cardinality of* $\mathcal{N}_{\mathbf{b}}(M)$ *by* $\#\mathcal{N}_{\mathbf{b}}(M)$*.*

We now prove the following technical lemma:

Lemma 2.13. *Let* $M \in \mathcal{M}_{\mathbf{b}}(\mathbf{a},\mathbf{e},r)$*. The cardinality* $\#\mathcal{N}_{\mathbf{b}}(M)$ *is completely determined by the elementary divisor form* $p^{\mathbf{e}}$ *of* M*, and the* n*-tuples* $\mathbf{b}$ *and* $\mathbf{a}$*. Hence we call this number* $\#\mathcal{N}_{\mathbf{b}}(\mathbf{a},\mathbf{e},r)$*.*

Proof. Let $M \in \mathcal{M} = \mathcal{M}_{\mathbf{b}}(\mathbf{a},\mathbf{e},r)$. We show that $\#\mathcal{N}_{\mathbf{b}}(M) = \#\mathcal{N}_{\mathbf{b}}(p^{\mathbf{e}})$.

Note that for $U \in G = GL_n(\mathbb{Z})$, $\begin{pmatrix} U^* & 0 \\ 0 & U \end{pmatrix} \in \Gamma = Sp_n(\mathbb{Z})$. Let $M = Up^{\mathbf{e}}V$ for some $U, V \in G$. Then since

$$\begin{pmatrix} U^* & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} p^r M^* & N \\ 0 & M \end{pmatrix} \begin{pmatrix} V^* & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} p^{r\mathbf{1}-\mathbf{e}} & U^*NV \\ 0 & p^{\mathbf{e}} \end{pmatrix}$$

we have that

$$\begin{pmatrix} p^r M^* & N \\ 0 & M \end{pmatrix} \sim_\Gamma \begin{pmatrix} p^{r\mathbf{1}-\mathbf{e}} & U^*NV \\ 0 & p^{\mathbf{e}} \end{pmatrix}.$$

We define $\phi : \mathcal{N}(M) \to \mathcal{N}(p^{\mathbf{e}})$ by the rule $\phi(N) = {}^tUNV^{-1}$ and show that ϕ is a bijection. First, we show that $\phi(N) \in \mathcal{N}(p^{\mathbf{e}})$. To show this we show ${}^t({}^tUNV^{-1})p^{\mathbf{e}} = p^{\mathbf{e}}({}^tUNV^{-1})$: Since $N \in \mathcal{N}(M)$, we have ${}^tNM = {}^tMN$. So, multiplying both sides of the equation by the same matrices we get $V^*({}^tNM)V^{-1} = V^*({}^tMN)V^{-1}$. Noting that $U^*\,{}^tU = I_n$ and $UU^{-1} = I_n$ we get $V^*\,{}^tN(UU^{-1})MV^{-1} = V^*\,{}^tM(U^*\,{}^tU)NV^{-1}$. Because $U^{-1}MV^{-1} = p^{\mathbf{e}}$ we have $(V^*\,{}^tNU)p^{\mathbf{e}} = {}^tp^{\mathbf{e}}({}^tUNV^{-1})$ whence ${}^t({}^tUNV^{-1})p^{\mathbf{e}} = p^{\mathbf{e}}({}^tUNV^{-1})$.

Second, ϕ is an injection for if $\phi(N) = \phi(N')$ then ${}^tUNV^{-1} = {}^tUN'V^{-1}$ which obviously implies that $N = N'$ since U and V are invertible.

Third, we show that ϕ is a surjection. Let $R \in \mathcal{N}(p^{\mathbf{e}})$. then we show $U^*RV \in \mathcal{N}(M)$: Since $R \in \mathcal{N}(p^{\mathbf{e}})$, ${}^tRp^{\mathbf{e}} = {}^tp^{\mathbf{e}}R$. Multiplying both sides of the equations by the same matrices gives us ${}^tV\,{}^tRp^{\mathbf{e}}V = {}^tVp^{\mathbf{e}}RV$. As $U^{-1}U = I_n$ and ${}^tUU^* = I_n$ we have ${}^tV\,{}^tR(U^{-1}U)p^{\mathbf{e}}V = {}^tVp^{\mathbf{e}}({}^tUU^*)RV$. Since $M = Up^{\mathbf{e}}V$ we get ${}^tV\,{}^tRU^{-1}M = {}^tMU^*RV$. By properties of the transpose we rewrite the last equation as ${}^t(U^*RV)M = {}^tM(U^*RV)$ and deduce that $U^*RV \in \mathcal{N}(M)$. Since $\phi\,(U^*RV) = R$ we conclude that ϕ is a surjection and hence, because of what was shown above, ϕ is a bijection.

Since

$$\begin{pmatrix} p^r M^* & N \\ 0 & M \end{pmatrix} \sim_\Gamma \begin{pmatrix} p^{r\mathbf{1}-\mathbf{e}} & U^*NV \\ 0 & p^{\mathbf{e}} \end{pmatrix}$$

they have the same symplectic divisor form by Proposition 2.4. Hence the number of $\left(\begin{smallmatrix} p^r M^* & N \\ 0 & M \end{smallmatrix}\right)$ and of $\left(\begin{smallmatrix} p^{r\mathbf{1}-\mathbf{e}} & U^*NV \\ 0 & p^{\mathbf{e}} \end{smallmatrix}\right)$ that have a particular symplectic divisor form, say $\left(\begin{smallmatrix} p^{r\mathbf{1}-\mathbf{b}} & 0 \\ 0 & p^{\mathbf{b}} \end{smallmatrix}\right)$, are the same. Note that

$$\left\{\left(\begin{smallmatrix} p^{r\mathbf{1}-\mathbf{e}} & U^*NV \\ 0 & p^{\mathbf{e}} \end{smallmatrix}\right)\right\}_{N\in\mathcal{N}(M)}$$

and

$$\left\{\left(\begin{smallmatrix} p^{r\mathbf{1}-\mathbf{e}} & N' \\ 0 & p^{\mathbf{e}} \end{smallmatrix}\right)\right\}_{N'\in\mathcal{N}(p^{\mathbf{e}})}$$

are equal as sets; in particular by Lemma 3.3.33 of [1], the upper right hand block of the matrices in these sets belong to:

$$\Big\{(b_{ij}) \in M_n(\mathbb{Z}) : b_{ij} < e_j (\text{for } 1 \le i \le j \le n) \text{ and } b_{ji} = ee_j^{-1} (\text{for } 1 \le i < j \le n)\Big\} \tag{2.2}$$

where $e_1, \ldots, e_n$ are the elementary divisors of M.

Because the map ϕ is a bijection, we have

$$\mathcal{N}_{\mathbf{b}}(M) = \phi^{-1}\left(\{U^*NV : U^*NV \in \mathcal{N}_{\mathbf{b}}(p^{\mathbf{e}})\}\right)$$

and hence $\#\mathcal{N}_{\mathbf{b}}(M) = \#\mathcal{N}_{\mathbf{b}}(p^{\mathbf{e}})$. □

We have previously described the actions of $\mathfrak{S}_n$ and W_n on polynomials. For a polynomial f, denote by $\operatorname{Stab}_{\mathfrak{S}_n}(f)$ or $\operatorname{Stab}_{W_n}(f)$ the appropriate stabilizer subgroup. We make the following definition:

Definition 2.14. *The symmetrized polynomial of $f \in \mathbb{Q}[x_1, \ldots, x_n]$ with respect to $\mathfrak{S}_n$, $\operatorname{Sym}_{\mathfrak{S}_n}(f)$, is defined to be*

$$\operatorname{Sym}_{\mathfrak{S}_n}(f) = \sum_{\sigma \in \mathfrak{S}_n / \operatorname{Stab}_{\mathfrak{S}_n}(f)} \sigma(f).$$

By convention, if f is constant, we let $\operatorname{Sym}_{\mathfrak{S}_n}(f) = f$.

The symmetrized polynomial of $g \in \mathbb{Q}[x_0, \ldots, x_n]$ with respect to W_n, $\operatorname{Sym}_{W_n}(g)$, is defined to be

$$\operatorname{Sym}_{W_n}(g) = \sum_{\sigma \in W_n / \operatorname{Stab}_{W_n}(g)} \sigma(g).$$

By convention, if g is constant, we let $\operatorname{Sym}_{W_n}(g) = g$.

Recall that for $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ we denote $x_1^{a_1} \cdots x_n^{a_n}$ by $x^{\mathbf{a}}$. Using Lemma 2.13 and the preceding definition, we compute the image of the Satake map on a double coset.

Proposition 2.15. *Let $r \ge 0$ and $\mathbf{b} \in \mathcal{C}_n(r)$. Then the image of the double coset $\Gamma \left(\begin{smallmatrix} p^{r\mathbf{1}-\mathbf{b}} & 0 \\ 0 & p^{\mathbf{b}} \end{smallmatrix}\right) \Gamma$ under the Satake map Ω has the form*

$$\Omega\left(\Gamma \left(\begin{smallmatrix} p^{r\mathbf{1}-\mathbf{b}} & 0 \\ 0 & p^{\mathbf{b}} \end{smallmatrix}\right) \Gamma\right) = x_0^r \sum_{\mathbf{a} \le \mathbf{b}} c(\mathbf{a}) \operatorname{Sym}_{\mathfrak{S}_n}(x^{\mathbf{a}})$$

for explicitly computable constants $c(\mathbf{a})$ where $\mathbf{a} \in \mathcal{C}_n(r)$.

Proof. We know that the image of $\Gamma\left(\begin{smallmatrix} p^{r\mathbf{1}-\mathbf{b}} & 0 \\ 0 & p^{\mathbf{b}}\end{smallmatrix}\right)\Gamma$ under Ω is invariant under the action of W_n, and in particular, by Proposition 2.10 we know that every right coset in the decomposition has similitude p^r. Thus, by Definition 2.7 and Lemma 2.9,

$$\Omega\left(\Gamma\left(\begin{smallmatrix} p^{r\mathbf{1}-\mathbf{b}} & 0 \\ 0 & p^{\mathbf{b}}\end{smallmatrix}\right)\Gamma\right) = x_0^r g(x_1,\dots,x_n) \tag{2.3}$$

where $g \in \mathbb{Q}[x_1,\dots,x_n]^{\mathfrak{S}_n}$, the ring of polynomials invariant under the action of $\mathfrak{S}_n$. So, if $c((a_1,\dots,a_n))x_1^{a_1}\cdots x_n^{a_n} := c(\mathbf{a})x^{\mathbf{a}}$ is a summand of g, then for every $\sigma \in \mathfrak{S}_n$ the monomial $c(\mathbf{a})x_1^{a_{\sigma(1)}}\cdots x_n^{a_{\sigma(n)}}$ is also a summand of g. By Proposition 2.10, it follows that to understand the image of Ω, we can limit our attention to those representatives $\Gamma\alpha = \Gamma\left(\begin{smallmatrix} p^r M^* & N \\ 0 & M\end{smallmatrix}\right)$ as described above for which the diagonal entries are p^a $(1 \le i \le n)$ for $\mathbf{a} = (a_1,\dots,a_n) \in \mathcal{C}_n(r)$. Thus for each $\mathbf{a} \in \mathcal{C}_n(r)$, $\mathbf{a} \le \mathbf{b}$, we find the coefficient $c(\mathbf{a})$ of $x^{\mathbf{a}}$, symmetrize the polynomial and multiply the new $\mathfrak{S}_n$-invariant polynomial by the coefficient $c(\mathbf{a})$ and the factor x_0^r to arrive at the image.

In (2.3), a summand $x^{\mathbf{a}}$ in g, comes from the right cosets of the form $\Gamma\alpha = \Gamma\left(\begin{smallmatrix} p^r M^* & N \\ 0 & M\end{smallmatrix}\right)$ where $M = \begin{pmatrix} p^{a_1} & * & \cdots & * \\ 0 & p^{a_2} & \cdots & * \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \cdots & p^{a_n}\end{pmatrix}$. We parse this collection of right cosets according to the elementary divisor form of M. The number of representatives with elementary divisor form $p^{\mathbf{e}}$ is given by $\#\mathcal{M}_{\mathbf{b}}(\mathbf{a},\mathbf{e},r)$. By Lemma 2.13, we see that for each $M \in \mathcal{M}$ there are $\#\mathcal{N}_{\mathbf{b}}(\mathbf{a},\mathbf{e},r)$ corresponding N. So by Corollary 2.11, the total number of representatives for each $\mathbf{a} \le \mathbf{b}$ is

$$\sum_{\mathbf{a}\le\mathbf{e}\le\mathbf{b}} \#\mathcal{M}_{\mathbf{b}}(\mathbf{a},\mathbf{e},r)\cdot\#\mathcal{N}_{\mathbf{b}}(\mathbf{a},\mathbf{e},r).$$

So, $c(\mathbf{a})$, the coefficient of $x^{\mathbf{a}}$ in g and hence in the Ω-image of $\Gamma\left(\begin{smallmatrix} p^{r\mathbf{1}-\mathbf{b}} & 0 \\ 0 & p^{\mathbf{b}}\end{smallmatrix}\right)\Gamma$ is

$$c(\mathbf{a}) = p^{-\sum i a_i}\left(\sum_{\mathbf{a}\le\mathbf{e}\le\mathbf{b}} \#\mathcal{M}_{\mathbf{b}}(\mathbf{a},\mathbf{e},r)\cdot\#\mathcal{N}_{\mathbf{b}}(\mathbf{a},\mathbf{e},r)\right). \tag{2.4}$$

As mentioned above, the coefficient of $x^{\mathbf{a}}$ must be the same as the coefficient of $x^{\mathbf{a}_\sigma}$ where $\mathbf{a}_\sigma = (a_{\sigma(1)},\dots,a_{\sigma(n)})$ for $\sigma^{-1} \in \mathfrak{S}_n/\operatorname{Stab}_{\mathfrak{S}_n}(x^{\mathbf{a}})$. Then, since by Proposition 2.10 all such $\mathbf{a}$ satisfy $\mathbf{a} \le \mathbf{b}$, we get

$$\begin{aligned}\Omega\left(\Gamma\left(\begin{smallmatrix} p^{r\mathbf{1}-\mathbf{b}} & 0 \\ 0 & p^{\mathbf{b}}\end{smallmatrix}\right)\Gamma\right) &= x_0^r\sum_{\mathbf{a}\le\mathbf{b}} c(\mathbf{a})\sum_{\sigma^{-1}\in\mathfrak{S}_n/\operatorname{Stab}_{\mathfrak{S}_n}(x^{\mathbf{a}})} x^{\mathbf{a}_\sigma} \\ &= x_0^r\sum_{\mathbf{a}\le\mathbf{b}} c(\mathbf{a})\operatorname{Sym}_{\mathfrak{S}_n}(x^{\mathbf{a}}).\end{aligned}$$

□

Remark 2.16. *We also comment that for $r = 2$ and any genus, [5] tells how to compute the coefficients in* (2.4) *explicitly; our computations agree with those in [5] modulo the weighting factor of the $|_k$ operator.*

By Ω_r we denote the restriction of Ω to $\underline{H}_p(\Gamma, S_p(r))$. We have just established that the codomain of Ω_r is the set $\mathbb{Q}_r[x_0, \ldots, x_n]^{W_n}$. Our goal is to show that the matrix representation of Ω_r ($r \in \mathbb{Z}, r \geq 0$) is upper-triangular and square.

A natural basis for $\underline{H}_p(\Gamma, S_p(r))$ consists of the distinct symplectic divisor forms (Proposition 2.2) of integral matrices of similitude p^r. That is, the basis consist of double cosets $\Gamma \left(\begin{smallmatrix} p^{r\mathbf{1}-\mathbf{b}} & 0 \\ 0 & p^{\mathbf{b}} \end{smallmatrix} \right) \Gamma$ for $\mathbf{b}$ running over the set

$$\mathcal{D} = \{(b_1, \ldots, b_n) \in \mathbb{Z}^n \mid r \geq b_1 \geq \cdots \geq b_n \geq r - b_n \geq \cdots \geq r - b_1 \geq 0\}.$$

More succinctly,

$$\mathcal{D} = \{(b_1, \ldots, b_n) \in \mathcal{C}_n(r) \mid b_n \geq r - b_n\}.$$

We shall show that there is a basis for $\mathbb{Q}_r[x_0, \ldots, x_n]^{W_n}$ naturally indexed by the same set $\mathcal{D}$, which in particular will show that the matrix $[\Omega_r]$ is square.

Definition 2.17. *We put an equivalence relation on $\mathcal{C}_n(r)$ as follows. Let $\mathbf{a}, \mathbf{b} \in \mathcal{C}_n(r)$. Write $a \cong b$ if $x_0^r x^{\mathbf{a}} \in \mathrm{Orbit}_{W_n}\left(x_0^r x^{\mathbf{b}}\right)$. Denote by $\mathcal{P}$ the set of equivalence classes in $\mathcal{C}_n(r)$.*

Lemma 2.18. *$\mathcal{P}$ and $\mathcal{D}$ have the same cardinality.*

Proof. Let $\overline{\mathbf{b}}$ denote the class of $\mathbf{b}$ in $\mathcal{P}$. We define the map $\psi : \mathcal{P} \to \mathcal{D}$ so that $\psi\left(\overline{\mathbf{b}}\right)$ is the maximal element in $\overline{\mathbf{b}}$ with respect to the lexicographic order $\leq$. We show that ψ is an injection. We also define the map $\phi : \mathcal{D} \to \mathcal{P}$ given by the rule that $\phi(\mathbf{b}) = \overline{\mathbf{b}}$ and show that it is an injection. As the two sets are finite, the result will follow.

First, we note that the set $\mathcal{C}_n(r)$ is totally ordered by $\leq$ and thus each $\overline{\mathbf{b}}$ has a unique maximal element denoted $\mathbf{b}_0$, so ψ is well-defined. Second, we show that ψ takes values in $\mathcal{D}$. If $\psi\left(\overline{\mathbf{b}}\right) = \mathbf{b}_0$, where $\mathbf{b}_0 = (b_1, \ldots, b_n)$ is the maximal element of $\overline{\mathbf{b}}$, then by definition, $\mathbf{b}_0 \in \mathcal{C}_n(r)$ and we need only show that $b_n \geq r - b_n$. Suppose $b_n < r - b_n$. Then $\mathbf{b}' = (b_1, \ldots, b_{n-1}, r - b_n) > \mathbf{b}_0 = (b_1, \ldots, b_{n-1}, b_n)$. But $\mathbf{b}_0 \cong \mathbf{b}'$ by Proposition 2.8, which contradicts the choice of $\mathbf{b}_0$ as the maximal element of $\overline{\mathbf{b}}$.

Third, we show that ψ is injective. Suppose that $\psi(\overline{\mathbf{b}}) = \psi(\overline{\mathbf{c}})$. Then both $\overline{\mathbf{b}}$ and $\overline{\mathbf{c}}$ have the same maximal element. Since $\cong$ is an equivalence relation and $\overline{\mathbf{b}}$ and $\overline{\mathbf{c}}$ have an element in common, we conclude $\overline{\mathbf{b}} = \overline{\mathbf{c}}$.

Now, we treat the map ϕ. Note that $\mathbf{b} = (b_1, \ldots, b_n) \in \mathcal{D}$ implies $b_1 \geq \cdots \geq b_n \geq r/2$. As the proof of the injectivity of ϕ is slightly more subtle when there is an $\frac{r}{2}$ among the b_i, we assume, without loss of generality, that $\mathbf{b}$ has the form $(b_1, \ldots, b_k, \frac{r}{2}, \ldots, \frac{r}{2})$ where $1 \leq k \leq n$ and $b_k > r/2$; so if $k = n$ there are no $\frac{r}{2}$ entries in $\mathbf{b}$. Suppose that $\phi(\mathbf{b}) = \phi(\mathbf{c})$; then $\mathbf{b}$ and $\mathbf{c}$ are in the same equivalence class, so $\mathbf{c} = (c_1, \ldots, c_k, \frac{r}{2}, \ldots, \frac{r}{2})$ since the action of W_n leaves the $\frac{r}{2}$ entries unchanged. There are three cases: $\mathbf{b} = \mathbf{c}$, $\mathbf{c} < \mathbf{b}$ or $\mathbf{b} < \mathbf{c}$. Suppose $\mathbf{c} < \mathbf{b}$. Then, since $\mathbf{c}$ and $\mathbf{b}$ are in the same W_n-equivalence class, we know by Proposition 2.8 that for some $1 \leq j \leq k$, $c_j = r - b_i$ for some $1 \leq i \leq k$. But, since, $c_j = r - b_i < \frac{r}{2}$ we get $\mathbf{c} \notin \mathcal{D}$. By symmetry, $\mathbf{b} < \mathbf{c}$ leads to a contradiction. Thus, $\mathbf{b} = \mathbf{c}$.

Since ϕ and ψ are both injections and the sets $\mathcal{P}$ and $\mathcal{D}$ are finite, they have the same cardinality. $\square$

Using the map ψ from Lemma 2.18, we can in fact sharpen the relationship between $\mathcal{D}$ and $\mathcal{P}$.

Proposition 2.19. *Let $\mathcal{P}'$ denote the set of maximum representatives of the equivalence classes in $\mathcal{P}$. With notation as above, $\mathcal{D} = \mathcal{P}'$.*

Proof. Let $\mathbf{b}_0 \in \mathcal{P}'$. With ψ as defined in the proof of Lemma 2.18 we note that $\psi\left(\overline{\mathbf{b}_0}\right) \in \mathcal{D}$. But, by the definition of $\mathcal{P}'$, we see that $\psi\left(\overline{\mathbf{b}_0}\right) = \mathbf{b}_0 \in \mathcal{D}$. Thus $\mathcal{P}' \subseteq \mathcal{D}$. By Lemma 2.18 $\mathcal{P}$ and $\mathcal{D}$ have the same finite cardinality; in particular, $\mathcal{P}'$ and $\mathcal{D}$ have the same finite cardinality. Thus $\mathcal{P}' = \mathcal{D}$. $\square$

It is now clear from the above discussion that there are bases of $\underline{H}_p(\Gamma, S_p(r))$ and $\mathbb{Q}_r[x_0, \ldots, x_n]^{W_n}$ which can be naturally indexed by the same set, and hence the spaces they span have the same dimension. More precisely:

Proposition 2.20. *Let $r \geq 0$ be an integer. The sets*

$$\mathcal{B}_1 = \left\{\Gamma \left(\begin{smallmatrix} p^{r\mathbf{1}-\mathbf{b}} & 0 \\ 0 & p^{\mathbf{b}} \end{smallmatrix}\right) \Gamma : \mathbf{b} \in \mathcal{D}\right\}$$

and

$$\mathcal{B}_2 = \left\{\mathrm{Sym}_{W_n}\left(x_0^r x^{\mathbf{b}}\right) : \mathbf{b} \in \mathcal{D}\right\}$$

are bases for $\underline{H}_p(\Gamma, S_p(r))$ and $\mathbb{Q}_r[x_0, \ldots, x_n]^{W_n}$ respectively.

Now we can assemble the previous results to prove our main theorem concerning the Satake map.

Theorem 2.21. *The matrix representing the linear transformation $\Omega_r : \underline{H}_p(\Gamma, S_p(r)) \to \mathbb{Q}_r[x_0, \ldots, x_n]^{W_n}$ is square and upper triangular with respect to the bases $\mathcal{B}_1$ and $\mathcal{B}_2$, each ordered in ascending order with respect to the lexicographic order $\leq$. In particular, for $\mathbf{b} \in \mathcal{D}$,*

$$\Omega\left(\Gamma \left(\begin{smallmatrix} p^{r\mathbf{1}-\mathbf{b}} & 0 \\ 0 & p^{\mathbf{b}} \end{smallmatrix}\right) \Gamma\right) = \sum_{\mathbf{a}\in\mathcal{D}, \mathbf{a}\leq\mathbf{b}} c(\mathbf{a})\, \mathrm{Sym}_{W_n}\left(x_0^r x^{\mathbf{a}}\right)$$

where the constants $c(\mathbf{a})$ are determined by equation (2.4) in the proof of Proposition 2.15. Moreover, the diagonal entries are nonzero.

Proof. By Proposition 2.20, $\underline{H}_p(\Gamma, S_p(r))$ and $\mathbb{Q}_r[x_0, \ldots, x_n]^{W_n}$ have the same dimension and thus the matrix representation of Ω_r is square. Since the bases of both the domain and co-domain can be indexed by the same set $\mathcal{D}$, the sum is over $\mathcal{D}$ and by Proposition 2.15 we know that we need only look at $\mathbf{a} \leq \mathbf{b}$. The fact that $\mathbf{a} \leq \mathbf{b}$ forces $[\Omega_r]$ to be upper triangular. The diagonal entries are nonzero since the right coset $\Gamma \left(\begin{smallmatrix} p^{r\mathbf{1}-\mathbf{b}} & 0 \\ 0 & p^{\mathbf{b}} \end{smallmatrix}\right)$ always appears in the decomposition of $\Gamma \left(\begin{smallmatrix} p^{r\mathbf{1}-\mathbf{b}} & 0 \\ 0 & p^{\mathbf{b}} \end{smallmatrix}\right) \Gamma$. $\square$

2.7 Examples

In this section we compute the matrix representation of Ω_1 for any genus and the matrix representation of Ω_2 for genus 2. Once again, we note that while this particular computation could be done via [5], the computation here gives a good indications of our methods which generalize in a straightforward manner to higher genera/similitudes. After computing the associated matrices, we invert them which will be put to use in defining arithmetically distinguished Hecke operators in the last section of the paper.

2.7.1 [Genus n, Similitude p]:

By definition, $\underline{H}_p(\Gamma, S_p(1))$ is the span of all $\Gamma \left(\begin{smallmatrix} p^{\mathbf{1}-\mathbf{b}} & 0 \\ 0 & p^{\mathbf{b}} \end{smallmatrix}\right) \Gamma$. There is only one such double coset and it is the generator $T(p)$. Thus the matrix representation of Ω_1 is a 1×1 matrix. We note that

$$\mathcal{C}_n(1) = \{(\underbrace{1, \ldots, 1}_{i}, \underbrace{0, \ldots, 0}_{n-i}) : 0 \leq i \leq n\}$$

and that all these are equivalent mod $\cong$ (see Definition 2.17). Let $\mathbf{0} := (\underbrace{0, \ldots, 0}_{n})$. We compute $c(\mathbf{0})$ as $\mathbf{0}$ is the easiest member of $\overline{\mathbf{1}}$ with which to compute. It is easiest because according to Proposition 2.10 the right coset decomposition $\Gamma \left(\begin{smallmatrix} pM^* & N \\ 0 & M \end{smallmatrix}\right)$ is such that M is reduced. When we limit our attention to $\mathbf{0}$, the diagonal entries of M are all 1. Thus, the only M we need consider is $M = I_n$. By equation (2.2) the only N that corresponds to $M = I_n$ is the zero matrix. The right coset in the decomposition of

$$\Gamma \left(\begin{smallmatrix} p^{\mathbf{1}-\mathbf{b}} & 0 \\ 0 & p^{\mathbf{b}} \end{smallmatrix}\right) \Gamma$$

whose lower right hand block has ones along the diagonal is $\Gamma \left(\begin{smallmatrix} pI_n & 0 \\ 0 & I_n \end{smallmatrix}\right) := \Gamma\alpha$ and thus $d_k(\alpha) = 1$ for all $1 \leq k \leq n$. By Proposition 2.4, then, $\alpha \in \Gamma \left(\begin{smallmatrix} p^{\mathbf{1}-\mathbf{b}} & 0 \\ 0 & p^{\mathbf{b}} \end{smallmatrix}\right) \Gamma$. The weighting factor $p^{-\sum i a_i}$ in the definition of Ω in this case is 1. So, by Theorem 2.21

$$\Omega\left(\Gamma \left(\begin{smallmatrix} p^{\mathbf{1}-\mathbf{b}} & 0 \\ 0 & p^{\mathbf{b}} \end{smallmatrix}\right) \Gamma\right) = x_0^1 \operatorname{Sym}_{W_n}(x_1 \cdots x_n)$$

and thus $[\Omega_1] = (1)$.

2.7.2 [Genus 2, Similitude p^2]

Now we look at Ω_2 for $n = 2$, but note that the methods below can be used for arbitrary n and r.

We start by finding $\mathcal{C}_2(2) = \{(2,2), (2,1), (2,0), (1,1), (1,0), (0,0)\}$. Next, we partition $\mathcal{C}_2(2)$ according to the equivalence relation $\cong$ as described in the previous section (Definition

2.17). According to Proposition 2.8 the equivalence classes are

$$\begin{aligned}\overline{(2,2)} &= \{(0,0),(2,0),(2,2)\}\\ \overline{(2,1)} &= \{(1,0),(2,1)\}\\ \overline{(1,1)} &= \{(1,1)\}\,.\end{aligned}$$

In other words, with notation from Proposition 2.19, $\mathcal{P}' = \{(1,1),(2,1),(2,2)\}$.

For $r=2$ and $n=2$ there are three double cosets that span $\underline{H}_p(\Gamma, S_p(2))$:

$$\Gamma \begin{pmatrix} p&0&0&0\\0&p&0&0\\0&0&p&0\\0&0&0&p \end{pmatrix} \Gamma,\ \Gamma \begin{pmatrix} 1&0&0&0\\0&p&0&0\\0&0&p^2&0\\0&0&0&p \end{pmatrix} \Gamma \text{ and } \Gamma \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&p^2&0\\0&0&0&p^2 \end{pmatrix} \Gamma.$$

Now, for each $\mathbf{a} \in \mathcal{D} = \mathcal{P}'$, we compute how many associated right coset representatives appear in each double coset. By Proposition 2.10 the right cosets are of the form

$$\begin{pmatrix} p^2M^* & N\\ 0 & M\end{pmatrix} \text{ where } M = \begin{pmatrix} p^{a_1} & m_{12}\\ 0 & p^{a_2}\end{pmatrix}$$

and $\mathbf{a} = (a_1, a_2) \leq \mathbf{b}$ where $p^{\mathbf{b}}$ is the lower right hand block of the double coset to which the matrix in question belongs.

Case 1: Let $\mathbf{a} = (2,2)$. Since the image of Ω_2 is a W_2-invariant polynomial, we know that $c((0,0)) = c((2,2))$. Computing $c((0,0))$ is easier. By Proposition 2.10 we know the right coset representative associated to $(0,0)$ is

$$\begin{pmatrix} p^2M^* & N\\ 0 & M\end{pmatrix} \text{ where } M = \begin{pmatrix} 1&0\\0&1\end{pmatrix}.$$

By equation (2.2) we know $N = \left(\begin{smallmatrix} 0&0\\0&0\end{smallmatrix}\right)$. Since the first and second determinantal divisors of this representative are both 1, we know this belongs to the double coset $\Gamma \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&p^2&0\\0&0&0&p^2 \end{pmatrix} \Gamma$.

Case 2: Let $\mathbf{a} = (2,1)$. Here it is easier to compute $c((1,0))$ instead of $c((2,1))$. By Proposition 2.10, the right coset representatives associated to $(1,0)$ are

$$\begin{pmatrix} p^2M^* & N\\ 0 & M\end{pmatrix} \text{ where } M = \begin{pmatrix} p&0\\0&1\end{pmatrix}.$$

By equation (2.2) we know $N = \left(\begin{smallmatrix} a&0\\0&0\end{smallmatrix}\right)$ where $0 \leq a < p$. Note that the first determinantal divisor of this representative is $d_1 = 1$ and the second is the $d_2 = \gcd(a,p)$. There are two cases:

- $a = 0$: in this case the representative belongs to the double coset $\Gamma \begin{pmatrix} 1&0&0&0\\0&p&0&0\\0&0&p^2&0\\0&0&0&p \end{pmatrix} \Gamma$. This happens once.
- $a \neq 0$: in this case $d_1 = 1$ and $d_2 = 1$ so the representative belongs to the double coset $\Gamma \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&p^2&0\\0&0&0&p^2 \end{pmatrix} \Gamma$. This happens $p-1$ times.

Case 3: Finally suppose $\mathbf{a} = (1,1)$. By Proposition 2.10, the right coset representative associated to $(1,1)$ is

$$\begin{pmatrix} p^2M^* & N \\ 0 & M \end{pmatrix} \text{ where } M = \begin{pmatrix} p & a \\ 0 & p \end{pmatrix}$$

for $0 \leq a < p$. There are two cases: $a = 0$ or otherwise.

- $a = 0$: in this case, by equation (2.2), we have $N = \begin{pmatrix} b & c \\ c & d \end{pmatrix}$ where $0 \leq b, c, d < p$. In this case the first determinantal divisor $d_1 = \gcd(b, c, d, p)$ and the second determinantal divisor $d_2 = \gcd(pb, pc, pd, bd - c^2, p^2)$. When $b, c, d = 0$ the right coset representative belongs to $\Gamma \begin{pmatrix} p&0&0&0 \\ 0&p&0&0 \\ 0&0&p&0 \\ 0&0&0&p \end{pmatrix} \Gamma$. This happens once.

  When either $b \neq 0, c, d = 0$; $b, c = 0, d \neq 0$; or $b, c, d \neq 0, bd - c^2 \equiv 0 \mod p$ the right coset belongs to $\Gamma \begin{pmatrix} 1&0&0&0 \\ 0&p&0&0 \\ 0&0&p^2&0 \\ 0&0&0&p \end{pmatrix} \Gamma$. This happens $p^2 - 1$ times.

  Otherwise the right coset belongs to $\Gamma \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&p^2&0 \\ 0&0&0&p^2 \end{pmatrix} \Gamma$ and this happens $p^3 - p^2$ times.

- $a \neq 0$: by equation (2.2) we know that N is independent of the elementary divisor form of M and by Lemma 2.13 we know that the number of right cosets that belong to each double coset is determined by $\mathrm{ed}(M)$. When $a \neq 0$ we have $\mathrm{ed}(M) = \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix}$ and that $N = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$ where $0 \leq b < p^2$. For all such b we have that d_1 and d_2 are both 1. Hence we conclude that all these representatives are in $\Gamma \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&p^2&0 \\ 0&0&0&p^2 \end{pmatrix} \Gamma$. Since a is nonzero $p - 1$ times, there are $p^2(p-1)$ such representatives.

Thus by Theorem 2.21

$$\begin{aligned}\Omega_2(T_2^2(p^2)) = \Omega_2\left(\Gamma \begin{pmatrix} p&0&0&0 \\ 0&p&0&0 \\ 0&0&p&0 \\ 0&0&0&p \end{pmatrix} \Gamma\right) &= \sum_{\mathbf{a} \in \mathcal{D}, \mathbf{a} \leq (1,1)} c(\mathbf{a}) \operatorname{Sym}_{W_2}(x_0^2 x^{\mathbf{a}}) \\ &= \frac{1}{p^3} x_0^2 x_1 x_2.\end{aligned}$$

Similarly by Theorem 2.21,

$$\begin{aligned}\Omega_2(T_1^2(p^2)) = \Omega_2\left(\Gamma \begin{pmatrix} 1&0&0&0 \\ 0&p&0&0 \\ 0&0&p^2&0 \\ 0&0&0&p \end{pmatrix} \Gamma\right) &= \sum_{\mathbf{a} \in \mathcal{D}, \mathbf{a} \leq (2,1)} c(\mathbf{a}) \operatorname{Sym}_{W_2}(x_0^2 x^{\mathbf{a}}) \\ &= \frac{p^2 - 1}{p^3} x_0^2 x_1 x_2 + \frac{1}{p} \operatorname{Sym}_{W_2}(x_0^2 x^{(2,1)}).\end{aligned}$$

Finally, also by Theorem 2.21,

$$\begin{aligned}\Omega_2(T_0^2(p^2)) &= \Omega_2\left(\Gamma\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&p^2&0\\0&0&0&p^2\end{pmatrix}\Gamma\right)\\ &= \sum_{\mathbf{a}\in\mathcal{D},\mathbf{a}\le(2,2)} c(\mathbf{a})\,\mathrm{Sym}_{W_2}(x_0^2x^{\mathbf{a}})\\ &= \frac{2p-2}{p}x_0^2x_1x_2 + \frac{p-1}{p}\,\mathrm{Sym}_{W_2}(x_0^2x^{(2,1)}) + \mathrm{Sym}_{W_2}(x_0^2x^{(2,2)}).\end{aligned}$$

From these equations we deduce that for $n=2$ the matrix representation for Ω_2 is

$$[\Omega_2] = \begin{pmatrix}\frac{1}{p^3} & \frac{p^2-1}{p^3} & \frac{2p-2}{p}\\ 0 & \frac{1}{p} & \frac{p-1}{p}\\ 0 & 0 & 1\end{pmatrix},$$

and for later use we point out that

$$[\Omega_2]^{-1} = \begin{pmatrix}p^3 & -p(p^2-1) & -(p-1)(p^2+1)\\ 0 & p & 1-p\\ 0 & 0 & 1\end{pmatrix}.$$

3 Symmetric polynomials and Hecke Operators

Our goal in this section is to define $(n+1)$ families of Hecke operators, $t_k^n(p^\ell)$, (analogous to the $T_k^n(p^2)$, $T(p)$) which are arithmetically interesting.

A large part of the arithmetic interest arises by examining the generating functions $\sum_\ell t_k^n(p^\ell)v^\ell$. The series have sums which are highly structured rational functions. In particular, in two of the $(n+1)$ cases, $t_0^n(p^\ell)$ and $t_1^n(p^\ell)$, the associated rational functions correspond to the spinor and standard zeta functions. In the other cases, they are new expressions.

As we have suggested, we shall make the definitions of the new operators, not in the Hecke algebra (defined by double cosets), but in its (isomorphic) representation space, the ring of W_n-invariant polynomials. Doing so will produce zeta functions in which the variables $x_0,\dots,x_n$ correspond (via the Satake correspondence) to the Satake p-parameters associated to a generic Hecke eigenform.

To define our Hecke operators in the context of this polynomial ring we need a definition and simple proposition: For a nonnegative integer ℓ, define $h^r(\ell) = \sum_{\substack{\sum j_k=\ell\\ j_k\ge 0}} z_1^{j_1}z_2^{j_2}\cdots z_r^{j_r}$. Note that $h^r(\ell)$ is a symmetric polynomial in the r variables $z_1,\dots,z_r$, and in particular, $h^r(0)=1$ and $h^r(1)=z_1+\cdots+z_r$.

Proposition 3.1. *The generating series associated to the $h^r(\ell)$ satisfies*

$$\sum_{\ell\ge 0} h^r(\ell)u^\ell = [(1-uz_1)\cdots(1-uz_r)]^{-1}$$

Proof. This is essentially obvious:

$$\begin{aligned}[(1-uz_1)\cdots(1-uz_r)]^{-1} &= \left(\sum_{a_1\geq 0}(uz_1)^{a_1}\right)\cdots\left(\sum_{a_r\geq 0}(uz_r)^{a_r}\right)\\ &= \sum_{\ell\geq 0} u^{\ell}\cdot\Big[\sum_{\substack{\sum a_i=\ell\\ a_i\geq 0}} z_1^{a_1}\cdots z_r^{a_r}\Big]\end{aligned}$$

It is clear from the definitions above that the coefficient of u^{ℓ} in the given expression is $h^r(\ell)$. ☐

Next we need to use the above polynomial to create a W_n-invariant polynomial. The simplest examples are simply to fix a monomial and to sum its images under the action of W_n. The following is a special case of Proposition 2.8.

Lemma 3.2. *Under the action of W_n, we obtain the following orbits:*

1. $\mathrm{Orbit}_{W_n}(x_0) = \{x_0 x_1^{\varepsilon_1}\cdots x_n^{\varepsilon_n} \mid \varepsilon_i = 0,1\}$.
2. $\mathrm{Orbit}_{W_n}(x_1\cdots x_k) = \{x_{i_1}^{\delta_{i_1}}\cdots x_{i_k}^{\delta_{i_k}} \mid 1\leq i_1\leq\cdots\leq i_k\leq n, \delta_{i_j}=\pm 1\}$.

In particular, the orbits have size 2^n and $2^k\binom{n}{k}$ respectively.

Having determined these orbits, the following definitions become less mysterious. We start with $h^r(\ell)$ where r is the size of one of the above orbits and substitute for the variables z_i the elements in the orbit. Thus we define the families of Hecke operators:

$$t_0^n(p^{\ell}) = h^{2^n}(\ell)\Big|_{\substack{z_i\mapsto\sigma_i(x_0)\\ \sigma_i\in W_n/\operatorname{Stab}(x_0)}}.$$

and for $1\leq k\leq n$,

$$t_k^n(p^{\ell}) = h^{2^k\binom{n}{k}}(\ell)\Big|_{\substack{z_i\mapsto\sigma_i(x_1\cdots x_k)\\ \sigma_i\in W_n/\operatorname{Stab}(x_1\cdots x_k)}}.$$

In particular,

$$t_0^n(p) = \sum_{\varepsilon_i=0,1} x_0 x_1^{\varepsilon_1}\cdots x_n^{\varepsilon_n}. \quad (2^n \text{ summands})$$

and

$$t_k^n(p) = \sum_{\substack{1\leq i_1<\cdots<i_k\leq n\\ \delta_{i_j}=\pm 1}} x_{i_1}^{\delta_{i_1}}\cdots x_{i_k}^{\delta_{i_k}}. \quad \left(2^k\binom{n}{k} \text{ summands}\right)$$

We now examine their generating series.

Theorem 3.3. *The operators $t_k(p^\ell)$ have generating series which are rational functions of the form:*

$$\sum_{\ell\geq 0} t_0^n(p^\ell)v^\ell = \left[(1-x_0v)\prod_{m=1}^{n}\prod_{1\leq i_1<\cdots<i_m\leq n}(1-x_0x_{i_1}\cdots x_{i_m}v)\right]^{-1},$$

and for $1\leq k\leq n$,

$$\sum_{\ell\geq 0} t_k^n(p^\ell)v^\ell = \left[\prod_{1\leq i_1<\cdots<i_k\leq n,\ \delta_{i_j}=\pm 1}(1-x_{i_1}^{\delta_{i_1}}\cdots x_{i_k}^{\delta_{i_k}}v)\right]^{-1}.$$

Proof. The proof is immediate from Proposition 3.1 and the computation of orbits in Lemma 3.2. □

Remark 3.4. *1. For $k=0$, the expression clearly corresponds to the local factor of the spinor zeta function: $Z_{F,p}(v)=(1-\alpha_0v)\prod_{m=1}^{n}\prod_{1\leq i_1<\cdots<i_m\leq n}(1-\alpha_0\alpha_{i_1}\cdots\alpha_{i_m}v)$. When $k=1$, the expression is simply $\sum_{\ell\geq 0} t_1^n(p^\ell)v^\ell = \left[\prod_{m=1}^{n}(1-x_mv)(1-x_m^{-1}v)\right]^{-1}$ which (up to an initial "zeta" factor) corresponds to the local factor of the standard zeta function: $D_{F,p}(v)=(1-v)\prod_{m=1}^{n}(1-\alpha_mv)(1-\alpha_m^{-1}v)$.*

2. *Except for $k=0$ and $k=1$, the Hecke operators, $t_k^n(p^\ell)$, give rise to new "zeta" functions which may be of interest in the context of Siegel modular forms.*

3. *Finally, we note that for the case of $n=2$, Andrianov [1] defines a family of Hecke operators $T^2(p^\ell)$ whose images under the Satake map Ω (from H_p to $\mathbb{Q}[x_0^{\pm 1},x_1^{\pm 1},x_2^{\pm 1}]^{W_2}$) satisfy*

$$\sum_{\ell\geq 0}\Omega(T^2(p^\ell))v^\ell = \frac{(1-p^{-1}x_0^2x_1x_2v^2)}{(1-x_0v)(1-x_0x_1v)(1-x_0x_2v)(1-x_0x_1x_2v)},$$

The operators $t_0^2(p^\ell)$ have a generating function whose sum has the same denominator as $\Omega(T^2(p^\ell))$, but with numerator 1.

4 New Hecke Operators in Genus 2

Since the generating series $\sum_{\ell\geq 0} t_k^n(p^\ell)v^\ell$ has the form $q_k^n(v)^{-1}$ for a polynomial $q_k^n(v)$, the relation $q_k^n(v)\cdot\sum_{\ell\geq 0} t_k^n(p^\ell)v^\ell = 1$ prescribes recursion relations to the operators $t_k^n(p^\ell)$. Given these relations and the requisite base cases, one can define classical Hecke operators in terms of double cosets which will have exactly the same generating series by inverting the Satake isomorphism. We do this below. For analogous operators on GL_n, this has been done in [8].

4.1 Recursion Relations

Since we are restricting to the case $n = 2$, we lighten the notational load a bit by writing t_k instead of t_k^2 and writing q_k instead of q_k^2. We also note that in the case of $n = 2$ we have the happy coincidence that the degree (in v) of $q_k(v)$ is 4, independent of the value of k, so we write

$$q_k(v) = \sum_{j=0}^{4}(-1)^j \varphi_j^{(k)} v^j; \quad \varphi_0^{(k)} = 1 \text{ for all } k.$$

If we take $t_k(p^m) = 0$ for $m < 0$ (note $t_k(1) = 1$), then the relation $q_k(v) \cdot \sum_{\ell \geq 0} t_k(p^\ell)v^\ell = 1$ yields for all $\ell \geq 1$:

$$t_k(p^\ell) = \varphi_1^{(k)} t_k(p^{\ell-1}) - \varphi_2^{(k)} t_k(p^{\ell-2}) + \varphi_3^{(k)} t_k(p^{\ell-3}) - \varphi_4^{(k)} t_k(p^{\ell-4}),$$

where

$$\begin{aligned}
\varphi_1^{(k)} &= t_k(p) \\
\varphi_2^{(k)} &= \varphi_1^{(k)} t_k(p) - t_k(p^2) \\
\varphi_3^{(k)} &= \varphi_2^{(k)} t_k(p) - \varphi_1^{(k)} t_k(p^2) + t(p^3) \\
\varphi_4^{(k)} &= \varphi_3^{(k)} t_k(p) - \varphi_2^{(k)} t_k(p^2) + \varphi_1^{(k)} t(p^3) - t_k(p^4)
\end{aligned}$$

From Theorem 3.3, we deduce:

$$\begin{aligned}
q_0(v) &= (1 - x_0 v)(1 - x_0 x_1 v)(1 - x_0 x_2 v)(1 - x_0 x_1 x_2 v) \\
q_1(v) &= (1 - x_1 v)(1 - x_1^{-1} v)(1 - x_2 v)(1 - x_2^{-1} v) \\
q_2(v) &= (1 - x_1 x_2 v)(1 - x_1 x_2^{-1} v)(1 - x_1^{-1} x_2 v)(1 - x_1^{-1} x_2^{-1} v)
\end{aligned}$$

from which we can compute the $\varphi_j^{(k)}$'s explicitly. Writing Sym for Sym_{W_2} we have:

$$\begin{aligned}
\varphi_1^{(0)} &= x_0(x_1+1)(x_2+1) = \mathrm{Sym}(x_0x_1x_2)\\
\varphi_2^{(0)} &= x_0^2(x_1x_2+1)(x_1+x_2) + 2x_0^2x_1x_2 = \mathrm{Sym}(x_0^2x_1^2x_2) + 2\,\mathrm{Sym}(x_0^2x_1x_2)\\
\varphi_3^{(0)} &= x_0^2x_1x_2(x_0(x_1+1)(x_2+1)) = \mathrm{Sym}(x_0^2x_1x_2)\,\mathrm{Sym}(x_0x_1x_2)\\
\varphi_4^{(0)} &= (x_0^2x_1x_2)^2 = \mathrm{Sym}(x_0^2x_1x_2)^2
\end{aligned}$$

$$\begin{aligned}
\varphi_1^{(1)} &= x_1 + x_1^{-1} + x_2 + x_2^{-1} = \mathrm{Sym}(x_0^2x_1x_2)^{-1}\,\mathrm{Sym}(x_0^2x_1^2x_2)\\
\varphi_2^{(1)} &= 2 + x_1x_2 + x_1x_2^{-1} + x_x^{-1}x_2 + x_1^{-1}x_2^{-1} = \mathrm{Sym}(x_0^2x_1x_2)^{-1}(2\,\mathrm{Sym}(x_0^2x_1x_2) + \mathrm{Sym}(x_0^2x_1^2x_2))\\
\varphi_3^{(1)} &= \varphi_1^{(1)}\\
\varphi_4^{(1)} &= 1
\end{aligned}$$

$$\begin{aligned}
\varphi_1^{(2)} &= x_1 + x_1^{-1} + x_2 + x_2^{-1} = \mathrm{Sym}(x_0^2x_1x_2)^{-1}\,\mathrm{Sym}(x_0^2x_1^2x_2)\\
\varphi_2^{(2)} &= 2 + x_1^2 + x_2^2 + x_1^{-2} + x_2^{-2}\\
&= -2 + \mathrm{Sym}(x_0^2x_1x_2)^{-2}(\mathrm{Sym}(x_0^2x_1^2x_2)^2 - 2\,\mathrm{Sym}(x_0^2x_1x_2)\,\mathrm{Sym}(x_0^2x_1^2x_2^2))\\
\varphi_3^{(2)} &= \varphi_1^{(2)}\\
\varphi_4^{(2)} &= 1
\end{aligned}$$

Remark 4.1. *Note that not all the expressions $\varphi_j^{(k)}$ belong to the integral polynomial ring $\mathbb{Q}[x_0,\dots,x_n]^{W_n}$, but rather to the Satake image of the full Hecke algebra: $\mathbb{Q}[x_0^{\pm1},\dots,x_n^{\pm1}]^{W_n} \cong \mathbb{Q}[x_0,\dots,x_n]^{W_n}[(x_0^2x_1\cdots x_n)^{-1}]$. While our method of inverting the Satake map is restricted to $\mathbb{Q}[x_0,\dots,x_n]^{W_n}$, we see that this identification of algebras allows us to invert any polynomial in $\mathbb{Q}[x_0^{\pm1},\dots,x_n^{\pm1}]^{W_n}$. In particular, we will see that when we invert our operators $t_k(p^\ell)$ some lie outside the ring of integral Hecke operators, suggesting perhaps a need to think more broadly about Hecke operators in trying to construct a Hecke theory as robust as in the elliptic modular case.*

Definition 4.2. *For $k = 0, 1, 2$, define*

$$\widetilde{T}_k(p^\ell) = \widetilde{T}_k^2(p^\ell) = \Omega^{-1}(t_k^2(p^\ell)).$$

Note that the $\widetilde{T}_k(p^\ell)$ are elements of the full Hecke algebra H_p for genus 2.

From the recursion relations deduced above:

$$t_k(p^\ell) = \varphi_1^{(k)}t_k(p^{\ell-1}) - \varphi_2^{(k)}t_k(p^{\ell-2}) + \varphi_3^{(k)}t_k(p^{\ell-3}) - \varphi_4^{(k)}t_k(p^{\ell-4}),$$

and using that Ω is a ring homomorphism, we see that

$$\widetilde{T}_k(p^\ell) = \Omega^{-1}(\varphi_1^{(k)})\widetilde{T}_k(p^{\ell-1}) - \Omega^{-1}(\varphi_2^{(k)})\widetilde{T}_k(p^{\ell-2}) + \Omega^{-1}(\varphi_3^{(k)})\widetilde{T}_k(p^{\ell-3}) - \Omega^{-1}(\varphi_4^{(k)})\widetilde{T}_k(p^{\ell-4}).$$

Noting that $t_0(p) = \varphi_1^{(0)}$, it is immediate from Example 2.7.1 that $\widetilde{T}_0(p) = T(p)$, one of the standard generators of the Hecke algebra. With the exception of $\varphi_1^{(0)}$, all the other $\varphi_j^{(k)}$ correspond to elements of $\mathbb{Q}[x_0, \ldots, x_n]^{W_n}$ of similitude 2, for which we can use Example 2.7.2 to help with the inversion.

In Example 2.7.2, the ordered basis for $\underline{H}_p(\Gamma, S_p(2))$ is $\mathcal{B} = \{T_2(p^2), T_1(p^2), T_0(p^2)\}$, while the ordered basis for $\mathbb{Q}_2[x_0, x_1, x_2]^{W_2}$ is $\mathcal{B}' = \{\mathrm{Sym}(x_0^2x_1x_2), \mathrm{Sym}(x_0^2x_1^2x_2), \mathrm{Sym}(x_0^2x_1^2x_2^2)\}$.

4.1.1 k=0

We have already shown that $\Omega^{-1}(\varphi_1^{(0)}) = \widetilde{T}_0(p) = T(p)$. Using the inverse computed in Example 2.7.2, we note that $\Omega^{-1}(\varphi_2^{(0)})$ corresponds to $[\Omega_2^{-1}]\,{}^t(2,1,0)_{\mathcal{B}'} = {}^t(2p^3 - p(p^2-1), p, 0)_{\mathcal{B}}$ where $(2,1,0)$ is the coordinate vector of $\varphi_2^{(0)}$ relative to the basis $\mathcal{B}'$. Thus $\Omega^{-1}(\varphi_2^{(0)}) = (p^3+p)T_2(p^2) + pT_1(p^2)$. For more complicated expressions, we first use that Ω is a ring homomorphism so that $\Omega^{-1}(\varphi_3^{(0)}) = \Omega^{-1}(\mathrm{Sym}(x_0^2x_1x_2)\,\mathrm{Sym}(x_0x_1x_2)) = \Omega^{-1}(\mathrm{Sym}(x_0^2x_1x_2))\Omega^{-1}(\mathrm{Sym}(x_0x_1x_2)) = p^3T_2(p^2)T(p)$. For completeness, $\Omega^{-1}(\varphi_4^{(0)}) = p^6T_2(p^2)^2$. Summarizing,

$$\Omega^{-1}(\varphi_1^{(0)}) = T(p) \qquad \Omega^{-1}(\varphi_2^{(0)}) = (p^3+p)T_2(p^2) + pT_1(p^2)$$
$$\Omega^{-1}(\varphi_3^{(0)}) = p^3T_2(p^2)T(p) \qquad \Omega^{-1}(\varphi_4^{(0)}) = p^6T_2(p^2)^2$$

and using

$$\widetilde{T}_k(p^\ell) = \Omega^{-1}(\varphi_1^{(k)})\widetilde{T}_k(p^{\ell-1}) - \Omega^{-1}(\varphi_2^{(k)})\widetilde{T}_k(p^{\ell-2}) + \Omega^{-1}(\varphi_3^{(k)})\widetilde{T}_k(p^{\ell-3}) - \Omega^{-1}(\varphi_4^{(k)})\widetilde{T}_k(p^{\ell-4}),$$

we can compute all $\widetilde{T}_0(p^\ell)$. In terms of the standard generators, the first few $\widetilde{T}_0(p^\ell)$ are given by:

$$\begin{aligned}
\widetilde{T}_0(p) &= T(p)\\
\widetilde{T}_0(p^2) &= T(p)^2 - [(p^3+p)T_2(p^2) + pT_1(p^2)]\\
\widetilde{T}_0(p^3) &= T(p)^3 - (p^3+2p)T(p)T_2(p^2) - 2pT(p)T_1(p^2)
\end{aligned}$$

4.1.2 k=1,2

Note that in the computation of $\varphi_j^{(k)}$ (when $k \geq 1$), we frequently need to contend with leaving the integral polynomial ring. As indicated before, this is not really an issue since $\mathbb{Q}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]^{W_n} \cong \mathbb{Q}[x_0, \ldots, x_n]^{W_n}[(x_0^2x_1\cdots x_n)^{-1}]$. So in particular, to find $\Omega^{-1}(\varphi_1^{(1)})$, we find

$$\begin{aligned}\widetilde{T}_1(p^2) = \Omega^{-1}(\varphi_1^{(1)}) &= \Omega^{-1}(\mathrm{Sym}(x_0^2x_1x_2)^{-1}\,\mathrm{Sym}(x_0^2x_1^2x_2)) \\ &= [\Omega^{-1}(\mathrm{Sym}(x_0^2x_1x_2))]^{-1}\Omega^{-1}(\mathrm{Sym}(x_0^2x_1^2x_2)) \\ &= p^{-3}T_2(p^2)^{-1}(-p(p^2-1)T_2(p^2)+pT_1(p^2)) \\ &= \frac{-p(p^2-1)}{p^3} + p^{-2}T_1(p^2)T_2(p^2)^{-1}\end{aligned}$$

is no longer in the integral Hecke algebra, $\underline{H}_p$, emphasizing the fact that not all Hecke operators with interesting arithmetic properties need to come from the integral Hecke ring.

Remark 4.3. *With some patience one can explore the action of these new operators on the Fourier coefficients of Siegel modular forms looking for recurrence relations among the Fourier coefficients as was done for the standard generators of the integral Hecke algebra (see [2]).*

References

[1] A. N. Andrianov, *Quadratic Forms and Hecke Operators*, Springer-Verlag, Berlin, 1987.

[2] Stefan Breulmann and Michael Kuss, *On a conjecture of Duke-Imamoḡlu*, Proc. Amer. Math. Soc. **128** (2000), no. 6, 1595–1604.

[3] P. Cartier, *Representations of p-adic groups: a survey*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Amer. Math. Soc., Providence, R.I., 1979, pp. 111–155.

[4] J. Hafner and L. Walling, *Siegel modular forms and Hecke operators in degree 2*, Millennial Conference on Number Theory, A K Peters, (2002).

[5] Aloys Krieg, *Das Vertauschungsgesetz zwischen Hecke-Operatoren und dem Siegelschen ϕ-Operator*, Arch. Math. (Basel) **46** (1986), no. 4, 323–329.

[6] Morris Newman, *Integral matrices*, Academic Press, New York, 1972, Pure and Applied Mathematics, Vol. 45.

[7] Cris Poor, personal communication.

[8] John A. Rhodes and Thomas R. Shemanske, *Rationality theorems for Hecke operators on GL_n*, J. Number Theory **102** (2003), 278 – 297.

[9] Goro Shimura, *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo, 1971, Kanô Memorial Lectures, No. 1.

[10] Tamara B. Veenstra, *Siegel modular forms, L-functions, and Satake parameters*, J. Number Theory **87** (2001), no. 1, 15–30.

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