Hecke Operators, Zeta Functions and the Satake map

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Abstract

Taking advantage of the Satake isomorphism, we define $(n+1)$ families of Hecke operators $t_k^n(p^\ell)$ for Sp_n whose generating series $\sum t_k^n(p^\ell)v^\ell$ are rational functions of the form $q_k(v)^{-1}$, where q_k is a polynomial in $\mathbb{Q}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}][v]$ of degree $2^k\binom{n}{k}$ (2^n if $k=0$). For $k=0$ and $k=1$ the form of the polynomial is essentially that of the local factors in the spinor and standard zeta functions. For $k>1$, these appear to be new expressions.

We also offer some insight (disjoint from the representation theory) for why there should be a correspondence between the local Hecke algebra and a ring of polynomials invariant under an associated Weyl group.

1 Introduction

Hecke theory for modular forms on the symplectic group is still very much in its infancy. Simplistically, the major stumbling block is that unlike the elliptic modular case, there is no obvious connection between the known invariants of the Hecke algebra (Satake p-parameters) and the Fourier coefficients of a Hecke eigenform, although there has been some interesting work done: using a partial knowledge of Satake parameters to infer complete knowledge ([8]), or finding correlations between Fourier coefficients and Hecke eigenvalues in degree 2 ([3]). Still we are very far away from a satisfactory general theory.

It is well-known (see e.g., Cartier [2], Theorem 4.1) that the Satake map shows that the p-part of the Hecke algebra associated to the symplectic group is isomorphic to a polynomial ring invariant under a certain Weyl group. In [1], Andrianov and Zhuravlev refer to this isomorphism as the spherical map, and give a description of it in terms of right cosets of the double cosets which generate the Hecke algebra.

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By working in the (isomorphic) representation space, we are able to define families of Hecke operators $t_k^n(p^\ell)$, $k = 0, \dots, n$ whose generating series have the form (see Theorem 3.3):

$$\sum_{\ell \geq 0} t_0^n(p^\ell) v^\ell = \left[(1 - x_0 v) \prod_{m=1}^{n} \prod_{1 \leq i_1 < \cdots < i_m \leq n} (1 - x_0 x_{i_1} \cdots x_{i_m} v) \right]^{-1}, \tag{1.1}$$

and for $1 \leq k \leq n$,

$$\sum_{\ell \geq 0} t_k^n(p^\ell) v^\ell = \left[\prod_{1 \leq i_1 < \cdots < i_k \leq n,\ \delta_{i_j} = \pm 1} (1 - x_{i_1}^{\delta_{i_1}} \cdots x_{i_k}^{\delta_{i_k}} v) \right]^{-1}. \tag{1.2}$$

To see the significance of these operators, recall that associated to a simultaneous Hecke eigenfunction F of weight k for $Sp_n(\mathbb{Z})$, are the Satake p-parameters $(\alpha_0, \dots, \alpha_n) = (\alpha_0(p), \dots, \alpha_n(p)) \in \mathbb{C}^{n+1}/W_n$ for each prime p (W_n the associated Weyl group), which generalize the Hecke eigenvalues. The Satake parameters satisfy $\alpha_0(p)^2 \alpha_1(p) \cdots \alpha_n(p) = p^{nk - n(n+1)/2}$ and are used to define the spinor and standard zeta functions.

The standard zeta function is defined by $D_F(s) = \prod_p D_{F,p}(p^{-s})^{-1}$ ($\Re(s) > 1$), where

$$D_{F,p}(v) = (1 - v) \prod_{m=1}^{n} (1 - \alpha_m v)(1 - \alpha_m^{-1} v),$$

while the spinor zeta function is defined by $Z_F(s) = \prod_p Z_{F,p}(p^{-s})^{-1}$ ($\Re(s) > nk/2 - n(n+1)/4 + 1$), where

$$Z_{F,p}(v) = (1 - \alpha_0 v) \prod_{m=1}^{n} \prod_{1 \leq i_1 < \cdots < i_m \leq n} (1 - \alpha_0 \alpha_{i_1} \cdots \alpha_{i_m} v).$$

For $k = 0$, the expression (1.1) clearly corresponds to the local factor of the spinor zeta function. When $k = 1$, the expression (1.2) is simply $\displaystyle\sum_{\ell \geq 0} t_1^n(p^\ell) v^\ell = \left[\prod_{m=1}^{n} (1 - x_m v)(1 - x_m^{-1} v) \right]^{-1}$ which (up to an initial "zeta" factor) corresponds to the local factor of the standard zeta function. Except for $k = 0$ and $k = 1$, the Hecke operators, $t_k^n(p^\ell)$, give rise to new "zeta" functions which may also be of interest in the context of Siegel modular forms. Independent of that fact, it is significant to have Hecke operators whose generating functions have this highly structured form. In subsequent work by Ryan [5], a local inverse for the Satake map is described, allowing these operators in the polynomial setting to be pulled back to classical Hecke operators in the symplectic setting, which are guaranteed to have generating functions which sum to rational functions of a highly structured form.

In the final section we make some remarks offering some intuition (not arising from the representation theory of p-adic groups) for why there should be a correspondence between the local Hecke algebra and a ring of symmetric polynomials.

2 The Classical Hecke Algebras

We shall deal with the Hecke algebra over $\mathbb{Q}$, and in particular with its local subalgebras. Much of this material can be found in Chapter 3 of [1]; we state it here to set the notation. Let $\Gamma = \Gamma_n = Sp_n(\mathbb{Z}) \subset SL_{2n}(\mathbb{Z})$, and let $G = GSp_n^+(\mathbb{Q}) \subset GL_{2n}(\mathbb{Q})$ be the group of symplectic similitudes with scalar factor $r(M) \in \mathbb{Q}_+^\times$:

$$\begin{aligned} GSp_n^+(\mathbb{Q}) &= \{M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in M_{2n}(\mathbb{Q}) \mid A^tC = C^tA, B^tD = D^tB, A^tD - C^tB = r(M)I_{2n}\} \\ &= \{M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in M_{2n}(\mathbb{Q}) \mid AB^t = B^tA, CD^t = DC^t, AD^t - BC^t = r(M)I_{2n}\}. \end{aligned}$$

Let H denote the rational Hecke algebra associated to the pair Γ and G. That is, as a vector space, H is generated by all double cosets $\Gamma\xi\Gamma$ ($\xi \in G$), and we turn H into an algebra by defining the multiplication law as follows: Given $\xi_1, \xi_2 \in G$, define

$$\Gamma\xi_1\Gamma \cdot \Gamma\xi_2\Gamma = \sum_{\xi} c(\xi)\Gamma\xi\Gamma, \tag{2.1}$$

where the sum is over all double cosets $\Gamma\xi\Gamma \subseteq \Gamma\xi_1\Gamma\xi_2\Gamma$, and the $c(\xi)$ are nonnegative integers (see [7]). There is an alternate characterization of the Hecke algebra which will be convenient as well. Let $L(\Gamma, G)$ be the rational vector space with basis consisting of right cosets $\Gamma\xi$ for $\xi \in G$. The Hecke algebra can be thought of as those elements of $L(\Gamma, G)$ which are right invariant under the action of Γ. Thus we can and will think of a double coset as the disjoint union of right cosets $\Gamma\xi\Gamma = \cup\Gamma\xi_\nu$ and as the sum of the same cosets $\sum \Gamma\xi_\nu \in L(\Gamma, G)$.

The global Hecke algebra, H, is generated by local Hecke algebras, H_p, one for each prime p, obtained as above by replacing G by $G \cap GL_{2n}(\mathbb{Z}[p^{-1}])$ in the above construction. H_p is generated by double cosets $\Gamma\xi\Gamma$ with ξ of the form $\mathrm{diag}(p^{a_1}, \ldots, p^{a_n}, p^{b_1}, \ldots, p^{b_n})$ where $a_1 \leq \cdots \leq a_n \leq b_n \leq \cdots \leq b_1$ are integers with $p^{a_i+b_i} = r(\xi)$ for all i. It is occasionally useful to consider the "integral" Hecke algebra $\underline{H}_p$ generated by all ξ as above with $\xi = \mathrm{diag}(p^{a_1}, \ldots, p^{a_n}, p^{b_1}, \ldots, p^{b_n}) \in M_{2n}(\mathbb{Z})$.

The integral Hecke algebra $\underline{H}_p$ is generated by the $(n+1)$ Hecke operators

$$T(p) = \Gamma \begin{pmatrix} I_n & 0 \\ 0 & pI_n \end{pmatrix} \Gamma$$

and for $k = 1, \ldots, n$,

$$T_k^n(p^2) = T_k(p^2) = \Gamma \begin{pmatrix} I_{n-k} & 0 & 0 & 0 \\ 0 & pI_k & 0 & 0 \\ 0 & 0 & p^2 I_{n-k} & 0 \\ 0 & 0 & 0 & pI_k \end{pmatrix} \Gamma,$$

while the Hecke algebra H_p is generated by the $(n+1)$ elements above together with the element $T_n(p^2)^{-1} = (pI_{2n})^{-1}$.

The Satake isomorphism (see [1]) shows that the local Hecke algebra is isomorphic to a polynomial ring invariant under a Weyl group:

$$\underline{H}_p \cong \mathbb{Q}[x_0, \ldots, x_n]^{W_n}$$
$$H_p \cong \mathbb{Q}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]^{W_n} \cong \mathbb{Q}[x_0, \ldots, x_n]^{W_n}[(x_0^2 x_1 \cdots x_n)^{-1}].$$

Here W_n is the group of $\mathbb{Q}$-automorphisms of the rational function field $\mathbb{Q}(x_0, \ldots, x_n)$ generated by all permutations of the variables $x_1, \ldots, x_n$ and by the automorphisms $\tau_1, \ldots, \tau_n$ which are given by:

$$\tau_i(x_0) = x_0 x_i, \qquad \tau_i(x_i) = x_i^{-1}, \qquad \tau_i(x_j) = x_j \quad (0 < j \neq i).$$

W_n is a signed permutation group, in particular, $W_n = \langle \tau_i \rangle \rtimes S_n \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n \cong C_n$ where C_n is Coxeter group associated to the spherical building for $Sp_n(\mathbb{Q}_p)$.

3 Symmetric polynomials and Hecke Operators

Our goal in this section is to define $(n+1)$ families of Hecke operators, $t_k^n(p^\ell)$, (analogous to the $T_k^n(p^2)$, $T(p)$) which are arithmetically interesting and at the same time naturally connected to the Bruhat-Tits building for $Sp_n(\mathbb{Q}_p)$.

A large part of the arithmetic interest arises by examining the generating functions $\sum_\ell t_k^n(p^\ell)v^\ell$. The series have sums which are highly structured rational functions. In particular, in two of the $(n+1)$ cases, $t_0^n(p^\ell)$ and $t_1^n(p^\ell)$, the associated rational functions correspond to the spinor and standard zeta functions. In the other cases, they are new expressions.

As we have suggested, we shall make the definitions of the new operators, not in the Hecke algebra (defined by double cosets), but in its (isomorphic) representation space, the ring of W_n-invariant polynomials. Doing so will produce zeta functions in which the variables $x_0, \ldots, x_n$ correspond (via the Satake correspondence) to the Satake p-parameters associated to a generic Hecke eigenform.

To define our Hecke operators in the context of this polynomial ring we need a definition and simple proposition: For a nonnegative integer ℓ, define $h^r(\ell) = \sum_{\substack{\sum j_k = \ell \\ j_k \geq 0}} z_1^{j_1} z_2^{j_2} \cdots z_r^{j_r}$. Note that $h^r(\ell)$ is a symmetric polynomial in the r variables $z_1, \ldots, z_r$, and in particular, $h^r(0) = 1$ and $h^r(1) = z_1 + \cdots + z_r$.

Proposition 3.1. *The generating series associated to the $h^r(\ell)$ satisfies*

$$\sum_{\ell \geq 0} h^r(\ell) u^\ell = [(1 - uz_1) \cdots (1 - uz_r)]^{-1}$$

Proof. This is essentially obvious:

$$
\begin{aligned}
[(1-uz_1)\cdots(1-uz_r)]^{-1} &= \left(\sum_{a_1\ge 0}(uz_1)^{a_1}\right)\cdots\left(\sum_{a_r\ge 0}(uz_r)^{a_r}\right)\\
&= \sum_{\ell\ge 0} u^\ell \cdot \Big[\sum_{\substack{\sum a_i=\ell\\ a_i\ge 0}} z_1^{a_1}\cdots z_r^{a_r}\Big]
\end{aligned}
$$

It is clear from the definitions above that the coefficient of u^ℓ in the given expression is $h^r(\ell)$. □

Next we need to use the above polynomial to create a W_n-invariant polynomial. The simplest examples are simply to fix a monomial and to sum its images under the action of W_n. To that end, we compute a few simple orbits.

Lemma 3.2. *Under the action of W_n, we obtain the following orbits:*

1. $\mathrm{Orbit}_{W_n}(x_0) = \{x_0 x_1^{\varepsilon_1}\cdots x_n^{\varepsilon_n} \mid \varepsilon_i = 0,1\}$.
2. $\mathrm{Orbit}_{W_n}(x_1\cdots x_k) = \{x_{i_1}^{\delta_{i_1}}\cdots x_{i_k}^{\delta_{i_k}} \mid 1\le i_1\le\cdots\le i_k\le n, \delta_{i_j}=\pm 1\}$.

In particular, the orbits have size 2^n and $2^k\binom{n}{k}$ respectively.

Proof. With the generators of W_n previously defined, we note that $\tau_{k_s}\tau_{k_{s-1}}\cdots\tau_{k_1}(x_0) = x_0x_{k_1}\cdots x_{k_s}$ for distinct $k_j\ge 1$, so it is clear that $\mathrm{Orbit}_{W_n}(x_0)\supseteq\{x_0x_1^{\varepsilon_1}\cdots x_n^{\varepsilon_n}\mid\varepsilon_i=0,1\}$, and hence the orbit has cardinality at least 2^n. On the other hand, all of S_n is contained in the stabilizer of x_0, so the size of the orbit is $[W_n : \mathrm{Stab}(x_0)] \le [W_n : S_n] = 2^n$, which gives the first result.

For the second, it is easy to see directly: S_n can take $x_1\cdots x_k$ to any monomial $x_{i_1}\cdots x_{i_k}$ with $1\le i_1\le\cdots\le i_k\le n$. Applying τ_{i_j} takes x_{i_j} to $x_{i_j}^{-1}$ fixing all other indices. Since these generate the group W_n, the orbit and its size are clear. □

Having determined these orbits, the following definitions become less mysterious. We start with $h^r(\ell)$ where r is the size of one of the above orbits and substitute for the variables z_i the elements in the orbit. Thus we define the families of Hecke operators:

$$t_0^n(p^\ell) = h^{2^n}(\ell)\Big|_{\substack{z_i\mapsto\sigma_i(x_0)\\ \sigma_i\in W_n/\mathrm{Stab}(x_0)}}.$$

and for $1\le k\le n$,

$$t_k^n(p^\ell) = h^{2^k\binom{n}{k}}(\ell)\Big|_{\substack{z_i\mapsto\sigma_i(x_1\cdots x_k)\\ \sigma_i\in W_n/\mathrm{Stab}(x_1\cdots x_k)}}.$$

In particular,

$$t_0^n(p) = \sum_{\varepsilon_i=0,1} x_0x_1^{\varepsilon_1}\cdots x_n^{\varepsilon_n}.\ \ (2^n \text{ summands})$$

and

$$t_k^n(p) = \sum_{\substack{1 \le i_1 < \cdots < i_k \le n \\ \delta_{i_j} = \pm 1}} x_{i_1}^{\delta_{i_1}} \cdots x_{i_k}^{\delta_{i_k}}. \quad (2^k \binom{n}{k} \text{ summands})$$

We now examine their generating series.

Theorem 3.3. *The operators $t_k(p^\ell)$ have generating series which are rational functions of the form:*

$$\sum_{\ell \ge 0} t_0^n(p^\ell) v^\ell = \left[(1 - x_0 v) \prod_{m=1}^{n} \prod_{1 \le i_1 < \cdots < i_m \le n} (1 - x_0 x_{i_1} \cdots x_{i_m} v) \right]^{-1},$$

and for $1 \le k \le n$,

$$\sum_{\ell \ge 0} t_k^n(p^\ell) v^\ell = \left[\prod_{1 \le i_1 < \cdots < i_k \le n,\ \delta_{i_j} = \pm 1} (1 - x_{i_1}^{\delta_{i_1}} \cdots x_{i_k}^{\delta_{i_k}} v) \right]^{-1}.$$

Proof. The proof is immediate from Proposition 3.1 and the computation of orbits in Lemma 3.2. □

Remark 3.4. *1. For $k = 0$, the expression clearly corresponds to the local factor of the spinor zeta function: $Z_{F,p}(v) = (1 - \alpha_0 v) \prod_{m=1}^{n} \prod_{1 \le i_1 < \cdots < i_m \le n} (1 - \alpha_0 \alpha_{i_1} \cdots \alpha_{i_m} v)$. When $k = 1$, the expression is simply $\sum_{\ell \ge 0} t_1^n(p^\ell) v^\ell = \left[\prod_{m=1}^{n} (1 - x_m v)(1 - x_m^{-1} v) \right]^{-1}$ which (up to an initial "zeta" factor) corresponds to the local factor of the standard zeta function: $D_{F,p}(v) = (1 - v) \prod_{m=1}^{n} (1 - \alpha_m v)(1 - \alpha_m^{-1} v)$.*

2. *Except for $k = 0$ and $k = 1$, the Hecke operators, $t_k^n(p^\ell)$, give rise to new "zeta" functions which may be of interest in the context of Siegel modular forms.*

3. *Finally, we note that for the case of $n = 2$, Andrianov and Zhuravlev [1] define a family of Hecke operators $T^2(p^\ell)$ whose images under the (Satake) spherical map Ω (from H_p to $\mathbb{Q}[x_0^{\pm 1}, x_1^{\pm 1}, x_2^{\pm 1}]^{W_2}$) satisfy*

$$\sum_{\ell \ge 0} \Omega(T^2(p^\ell)) v^\ell = \frac{(1 - p^{-1} x_0^2 x_1 x_2 v^2)}{(1 - x_0 v)(1 - x_0 x_1 v)(1 - x_0 x_2 v)(1 - x_0 x_1 x_2 v)},$$

The operators $t_0^2(p^\ell)$ have a generating function whose sum has the same denominator as $\Omega(T^2(p^\ell))$, but with numerator 1.

Remark 3.5. *Since the generating series $\sum_{\ell\geq 0} t_k^n(p^\ell)v^\ell$ has the form $q_k(v)^{-1}$ for a polynomial $q_k(v)$, the relation $q_k(v)\cdot\sum_{\ell\geq 0} t_k^n(p^\ell)v^\ell = 1$ prescribes recursion relations to the operators $t_k^n(p^\ell)$. Given these relations and the requisite base cases, one can define classical Hecke operators in terms of double cosets which will have exactly the same generating series by inverting the Satake isomorphism. For analogous operators on GL_n, this has been done in [4]. For Sp_n, this work has been done in [5].*

4 Connections to the Satake map

We have taken advatage of the isomorphism provided by the Satake map between the local Hecke algebra, H_p, and the ring of polynomials $\mathbb{Q}[x_0^{\pm1},\dots,x_n^{\pm1}]$ which are invariant under the Weyl group W_n, though this correspondence (at least at the level of cosets mapping to polynomials) is far from intuitive.

In this final section we give a labeling of the special vertices in an apartment of the Bruhat-Tits building for $Sp_n(\mathbb{Q}_p)$ by monomials in $\mathbb{Q}[x_0^{\pm1},\dots,x_n^{\pm1}]$ which corresponds in a natural way to the labeling of vertices in terms of a symplectic basis for the underlying space. This is turn [6] gives a correspondence with double cosets and hence a classical Hecke algebra which makes the connection between the Hecke algebra and polynomilas seem natural. On the other hand, it does not appear that this correspondence is the Satake map, but at least provides (another) intuition that there should be such a correspondence.

Actually, our labeling of vertices will be by elements in $\mathbb{Q}[x_0^{\pm1},\dots,x_n^{\pm1}]$ modulo the relation $x_0^2x_1\cdots x_n = 1$, which will mirror that the vertices are themselves labeled by homothety classes of lattices. This is also a very natural condition in terms of the Satake parameters. Recall that the variables x_i are playing the role of the Satake p-parameters $\alpha_i(p)$ which, for a simultaneous Hecke eigenform of weight k for $Sp_n(\mathbb{Z})$, satisfy $\alpha_0(p)^2\alpha_1(p)\cdots\alpha_n(p) = p^{nk-n(n+1)/2}$. Thus, modulo the power of p which is "invisible" at the level of an apartment, this is exactly the same condition. Finally, since $H_p \cong \mathbb{Q}[x_0^{\pm1},\dots,x_n^{\pm1}]^{W_n} \cong \mathbb{Q}[x_0,\dots,x_n]^{W_n}[(x_0^2x_1\cdots x_n)^{-1}]$ (see [1]), reducing by the relation $x_0^2x_1\cdots x_n = 1$ produces a subring of $\mathbb{Q}[x_0,x_1,\dots,x_n]^{W_n} \cong \underline{H}_p$, the integral local Hecke algebra.

Using the notation of [6], fix a (fundamental) apartment Σ in the building by means of a frame and symplectic basis $\{u_1,\dots,u_n,w_1,\dots,w_n\}$. Let $[\Lambda_0]$ be the class of the lattice $\Lambda_0 = \mathbb{Z}_pu_1\oplus\cdots\oplus\mathbb{Z}_pu_n\oplus\mathbb{Z}_pw_1\oplus\cdots\oplus\mathbb{Z}_pw_n$, labeling a fixed special vertex in the apartment Σ. In [6], we saw that a typical vertex $[\Lambda]$ in Σ is special iff the vertex is self-dual, that is $\Lambda = \mathbb{Z}_pp^{a_1}u_1\oplus\cdots\oplus\mathbb{Z}_pp^{a_n}u_n\oplus\mathbb{Z}_pp^{b_1}w_1\oplus\cdots\oplus\mathbb{Z}_pp^{b_n}w_n$ for which there is an integer μ with $\mu = a_i + b_i$ for all i. With this notation, we now have a one-to-one correspondence between the classes of lattices (labeling special vertices), and monomials in $\mathbb{Q}[x_0^{\pm1},\dots,x_n^{\pm1}]$ given by:

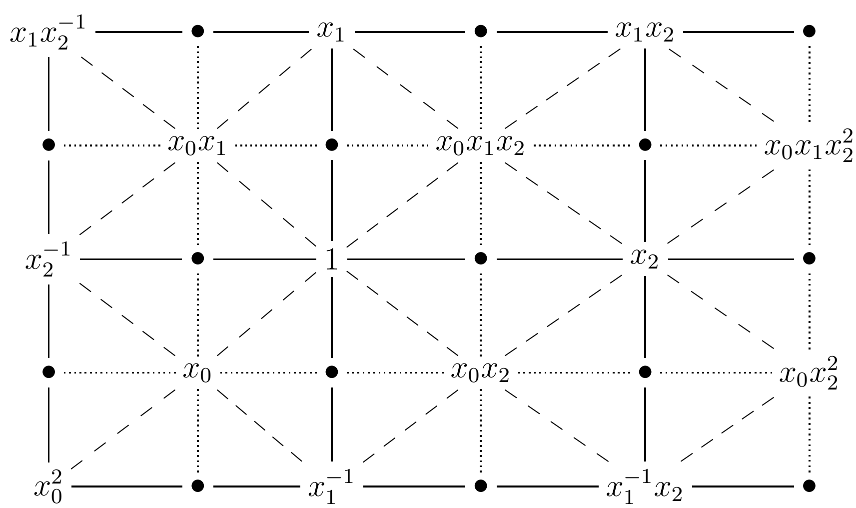
$$[p^{a_1},\dots,p^{a_n};p^{b_1},\dots,p^{b_n}] \longleftrightarrow x_0^\mu x_1^{a_1}\cdots x_n^{a_n}$$

modulo the relation $x_0^2x_1\cdots x_n = 1$ which corresponds to the class $[p,\dots,p;p,\dots,p] = [\Lambda_0]$. That is, if Λ is replaced by $p^c\Lambda$, then $x_0^\mu x_1^{a_1}\cdots x_n^{a_n}$ is replaced by $(x_0^2x_1\cdots x_n)^c x_0^\mu x_1^{a_1}\cdots x_n^{a_n}$, so that classes of lattices correspond to classes of monomials. To keep the notation from getting too involved, we will simply write $x_0^{\mu}x_1^{a_1}\cdots x_n^{a_n}$ rather than $[x_0^{\mu}x_1^{a_1}\cdots x_n^{a_n}]$. This avoids obvious confusion in statements like $\mathbb{Q}[x_0^{\pm 1},\ldots,x_n^{\pm 1}]^{W_n}=\mathbb{Q}[x_0,\ldots,x_n]^{W_n}[(x_0^2x_1\cdots x_n)^{-1}]$.

On the other hand, with the given notation, there is an obvious correspondence with the local Hecke algebra: Given, $[p^{a_1},\ldots,p^{a_n};p^{b_1},\ldots,p^{b_n}] \longleftrightarrow x_0^{\mu}x_1^{a_1}\cdots x_n^{a_n}$ with $\mu=a_i+b_i$, we immediately note that $\operatorname{diag}(p^{a_1},\ldots,p^{a_n};p^{b_1},\ldots,p^{b_n})\in GSp_n(\mathbb{Q}_p)$, so that $\Gamma\operatorname{diag}(p^{a_1},\ldots,p^{a_n};p^{b_1},\ldots,p^{b_n})\Gamma$ is in the local Hecke algebra H_p. Thus there is a clear connection between the Hecke operator $\Gamma\operatorname{diag}(p^{a_1},\ldots,p^{a_n};p^{b_1},\ldots,p^{b_n})\Gamma$ and the monomial $x_0^{\mu}x_1^{a_1}\cdots x_n^{a_n}$. Sadly, this correspondence does not appear to be a homomorphism, nonetheless it offers some motivation that there should be a natural correspondence.

Here we provide a labeling of a piece of the apartment Σ for Sp_2, corresponding to our previous labeling by classes of lattices (Example 2.4 of [6]):

Example 4.1. *A partial labeling of the special vertices in an apartment for $Sp_2(\mathbb{Q}_p)$ by (classes of) monomials*



Remark 4.2. *We make one final connection of these monomials to the Hecke algebra. We began this paper by defining polynomial Hecke operators $t_k^n(p^{\ell})$. Consider the generators $t_k^n(p)$ when $n=2$ so that we can use the above labelings given in Example 4.1 of this paper and Example 2.4 of [6].*

$$t_0^2(p)=x_0+x_0x_1+x_0x_2+x_0x_1x_2=[1,1;p,p]+[p,1;1,p]+[1,p;p,1]+[p,p;1,1]$$
$$t_1^2(p)=x_1+x_1^{-1}+x_2+x_2^{-1}=[p^2,p;1,p]+[1,p;p^2,p]+[p,p^2;p,1]+[p,1,;p,p^2]$$
$$t_2^2(p)=x_1x_2+x_1x_2^{-1}+x_1^{-1}x_2+x_1^{-1}x_2^{-1}=[p^2,p^2;1,1]+[p^2,1;1,p^2]+[1,p^2;p^2,1]+[1,1;p^2,p^2]$$

That is, the operator $t_k^n(p)$ is a formal sum of monomials. Viewed as a sum of classes of monomials, these sums correspond exactly to the sums over classes of lattices in the *fundamental apartment, and in particular again look like adjacency operators. Finally, note that the difference between the actual monomials and their classes is the same as the difference between the abstract Hecke algebra and its representation space acting on lattices or modular forms.*

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