Rationality Theorems for Hecke Operators on GL_n

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Abstract

We define n families of Hecke operators $T_k^n(p^\ell)$ for GL_n whose generating series $\sum T_k^n(p^\ell)u^\ell$ are rational functions of the form $q_k(u)^{-1}$ where q_k is a polynomial of degree $\binom{n}{k}$, and whose form is that of the kth exterior product. This work can be viewed as a refinement of work of Andrianov [1], in which he defined Hecke operators the sum of whose generating series was a rational function with nontrivial numerator and whose denominator was essentially $\prod_k q_k$.

By a careful analysis of the Satake map which defines an isomorphism between a local Hecke algebra and a ring of symmetric polynomials, we define n families of (polynomial) Hecke operators and characterize their generating series as rational functions. We then give an explicit means by which to locally invert the Satake isomorphism, and show how to translate these polynomial operators back to the classical double coset setting. The classical Hecke operators have generating series of exactly the same form as their polynomial counterparts, and hence are of number-theoretic interest. We give explicit examples for GL_3 and GL_4.

1 Introduction

While Hecke theory for automorphic forms on GL_2 is fairly robust, far less is known for GL_n when $n > 2$, especially as it relates to the arithmetic theory of modular forms. It is well-known (see e.g., Cartier [3], Theorem 4.1) that the Satake map shows that the p-part of the Hecke algebra associated to certain p-adic reductive groups is isomorphic to a polynomial ring invariant under an associated Weyl group. In [2], Andrianov and Zhuravlev refer to this isomorphism as the spherical map, and give a description of it for the general linear and symplectic groups in terms of right cosets of the double cosets which generate the Hecke algebra.

An important aspect of this isomorphism is that it allows one to think about Hecke operators in the polynomial algebra, a setting in which the multiplication is more straightforward

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than the usual product of double cosets. Our goals in this paper are to define families of Hecke operators in the polynomial algebra which have generating series whose sum is an extremely nice rational function, and then to invert the Satake isomorphism, to produce "classical" Hecke operators having generating series of exactly the same form, and hence of number-theoretic interest.

In [1], Andrianov considers classical Hecke operators for GL_n, and studies their generating series. He shows (Theorem 3) that the image of their sum under the spherical map is a rational function (in $\mathbb{Q}(x_1, \dots, x_n, u)$) with nontrivial numerator, and whose denominator is a product essentially of the form:

$$\prod_{k=1}^{n} \prod_{1\le i_1<i_2<\cdots<i_k\le n} (1 - x_{i_1}x_{i_2}\cdots x_{i_k}u)$$

Notice that the monomials which occur in the inner product, $x_{i_1}x_{i_2}\cdots x_{i_k}$, are kth exterior powers of the variables $x_1, \dots, x_n$.

In this paper, we take as a starting point the polynomial ring, and define families of symmetric polynomials, denoted $t_k^n(p^\ell)$ $(k = 1, \dots, n)$ and which we will call polynomial Hecke operators, whose associated generating series are rational functions. In particular (see Proposition 3.5)

$$\sum_{\ell\ge 0} t_k^n(p^\ell)u^\ell = \prod_{1\le i_1<i_2<\cdots<i_k\le n} (1 - x_{i_1}x_{i_2}\cdots x_{i_k}u)^{-1}.$$

Thus the operators $t_k(p^\ell)$ give a significant refinement to the operators investigated by Andrianov. In particular, we are producing a family of operators whose generating series are rational functions with trivial numerators, and whose denominators are the individual factors of Andrianov's product. Moreover, we shall see that one family, the $t_1^n(p^\ell)$, are essentially formal analogs of the classical Hecke operators $T(p^\ell)$.

The more difficult task is to determine to what combination of double cosets these polynomial operators correspond. To accomplish this one must invert the spherical map. By studying the restriction of the spherical map to those double cosets of a fixed determinant, it is possible to (locally) invert the spherical map, the matrix of the restricted map being upper triangular. At the heart of the computations to determine the matrix of the linear map is the question of determining the number of right cosets of a specified form which occur in the decomposition of a given double coset. In general, this is quite onerous, however we give a number of ways in which the computation can be significantly simplified.

Intertwined with the general theory, we provide examples which explicitly compute families $(T_k^4(p^\ell))$ of classical Hecke operators for GL_4. The interest in GL_4 is that $n = 4$ is the first case in which we associate "Euler factors" of degree greater than n with Hecke operators for GL_n. In particular, Proposition 3.5 shows that the denominator of the rational function associated to $\sum_{\ell\ge 0} t_k^n(p^\ell)u^\ell$ is the product of $\binom{n}{k}$ linear factors, thus producing Hecke operators for GL_n with associated Euler factors of degree $\binom{n}{k}$.

2 The Hecke Algebras

The global Hecke algebra described by Shimura [6] is generated as a rational vector space by double cosets $\Lambda\xi\Lambda$ with respect to the discrete group $\Lambda = SL_n(\mathbb{Z})$ and $\xi \in M_n^+(\mathbb{Z})$ (integer matrices with positive determinant), while the algebra described by Andrianov and Zhuravlev [2] is generated by double cosets $\Gamma\xi\Gamma$ with $\Gamma = GL_n(\mathbb{Z})$ and $\xi \in GL_n(\mathbb{Q})$. If we denote by $H(K, S)$ the rational Hecke algebra generated by double cosets $K\xi K$ where K is a subgroup of G, $S \supset K$ is a semigroup in the commensurator of K, and $\xi \in S$, then we have natural maps

$$H(SL_n(\mathbb{Z}), M_n^+(\mathbb{Z})) \to H(SL_n(\mathbb{Z}), GL_n^+(\mathbb{Q})) \to H(GL_n(\mathbb{Z}), GL_n(\mathbb{Q})),$$

where the map on the left is an injection, and that on the right an isomorphism. Moreover, a straightforward argument shows that

Proposition 2.1. *$H(GL_n(\mathbb{Z}), GL_n(\mathbb{Q}))$ is generated, as a ring, by $H(SL_n(\mathbb{Z}), M_n^+(\mathbb{Z}))$ together with the elements $GL_n(\mathbb{Z})(p^{-1}I_n)GL_n(\mathbb{Z})$ for all primes p. (I_n denotes the $n \times n$ identity matrix.)*

Thus for investigating Hecke theory, it makes little difference which algebra we choose. We adopt the notation of [2], and set the global Hecke algebra $H^n = H(GL_n(\mathbb{Z}), GL_n(\mathbb{Q}))$. By Theorems 2.3 and 2.8 of [2], H^n is commutative, and generated by its local subrings, H_p^n one for each prime p, defined by

$$H_p^n = H(GL_n(\mathbb{Z}), GL_n(\mathbb{Z}[p^{-1}])) \subset H^n.$$

The integral subring $\underline{H}_p^n$ of H_p^n is

$$\begin{aligned}\underline{H}_p^n &= \langle \Gamma\xi\Gamma \mid \xi \in M_n(\mathbb{Z}), \det(\xi) = \pm p^\lambda \rangle \\ &= \langle \Gamma \operatorname{diag}(p^{i_1}, \dots, p^{i_n})\Gamma \mid \lambda \geq i_1 \geq \cdots \geq i_n \geq 0 \rangle\end{aligned}$$

where the angle brackets enclose generators as a $\mathbb{Q}$-vector space. As a ring, H_p^n is generated by $\underline{H}_p^n$ together with the single element $\Gamma(p^{-1}I_n)\Gamma$. Thus the study of the the global Hecke ring H^n reduces to study of the local Hecke rings H_p^n, which in turn can be understood through their integral subrings $\underline{H}_p^n$.

Equivalently, we could focus on the p-adic Hecke ring. Letting $\Gamma_p = GL_n(\mathbb{Z}_p)$, the p-adic Hecke ring is

$$\mathcal{H}_p^n = H(GL_n(\mathbb{Z}_p), GL_n(\mathbb{Q}_p)),$$

and its integral subring

$$\underline{\mathcal{H}}_p^n = \langle \Gamma_p(\operatorname{diag}(p^{i_1}, \dots, p^{i_n}))\Gamma_p \mid i_1 \geq \cdots \geq i_n \geq 0 \rangle,$$

which together with the single element $\Gamma_p(p^{-1}I_n)\Gamma_p$ generates $\mathcal{H}_p^n$. Since $\mathcal{H}_p^n$ (resp. $\underline{\mathcal{H}}_p^n$) is canonically isomorphic to H_p^n (resp. $\underline{H}_p^n$), we could work in the p-adic setting (as in studying

p-adic reductive groups and buildings), but for our purposes here, we shall simply keep things local and not p-adic.

The integral Hecke algebra $\underline{H}_p$ is generated by the n Hecke operators

$$\pi_k^n(p) = \Gamma \operatorname{diag}(\underbrace{p, \ldots, p}_{k}, \underbrace{1, \ldots, 1}_{n-k})\Gamma, \quad k = 1, \ldots, n$$

and the local Hecke algebra H_p is generated by the n elements above, plus the element $\pi_n^n(p)^{-1}$ (see [2]). The Satake map is an isomorphism between the (p-adically defined) local Hecke algebra and a ring of symmetric polynomials. In [2], it is called the spherical map, and we will examine it carefully in §4. For now, simply note that if we let S_n denote the symmetric group on n letters and $\mathbb{Q}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n}$ the ring of symmetric polynomials in the variables $x_1^{\pm 1}, \ldots, x_n^{\pm 1}$ (i.e. those polynomials invariant under the natural action of the symmetric group on the variables), then the spherical map is denoted $\omega : H_p \to \mathbb{Q}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n}$, and takes $\underline{H}_p$ isomorphically onto $\mathbb{Q}[x_1, \ldots, x_n]^{S_n}$ (the usual ring of symmetric polynomials). In particular, it maps $\pi_k^n(p)$ to a power of p times the kth elementary symmetric polynomial (see [2]).

3 Polynomial Hecke Operators

As discussed in the introduction, the Satake isomorphism allows one to study the Hecke algebra in the context of a polynomial ring, with very familiar ring operations, instead of its defined setting where the multiplication of double cosets is cumbersome.

In this section, we shall define families of symmetric polynomials, denoted $t_k^n(p^\ell)$ ($k = 1, \ldots, n$), which we will call polynomial Hecke operators, whose associated generating series are rational functions of an especially nice form (see Proposition 3.5 below).

To begin our investigation, note that the symmetric group S_n acts naturally on polynomials in n variables. Thus for a polynomial p it makes sense to refer to the stabilizer $\operatorname{Stab}(p)$ in S_n of p. For a polynomial p in n variables, define the *symmetrized polynomial* associated to p, $\operatorname{Sym}_n(p)$, by

$$\operatorname{Sym}_n(p) = \sum_{\sigma \in S_n / \operatorname{Stab}(p)} \sigma(p)$$

We understand that if p is a constant, that $\operatorname{Sym}_n(p) = p$.

Example 3.1. *In* $\mathbb{Q}[z_1, \ldots, z_n]$, $\operatorname{Sym}_n(z_1 \cdots z_k) = \sum_{1 \le i_1 < \cdots < i_k \le n} z_{i_1} \cdots z_{i_k}$.

Let m and n be positive integers and denote by $P_n(m)$ the set of partitions of m into n pieces such that a given partition satisfies: $m \ge i_1 \ge \cdots \ge i_n \ge 0$ (and $\sum i_k = m$). Introduce the lexicographic ordering on $P_n(m)$, and let $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in P_n(m)$. For indeterminates $z_1, z_2, \ldots, z_n$ define

$$\begin{aligned} &hp(0,\ldots,0) = 1, \\ &hp(\mathbf{i}) = hp(i_1, i_2, \ldots, i_n) = \sum_{\substack{\mathbf{j} \le \mathbf{i} \\ \mathbf{j} \in P_n(m)}} \mathrm{Sym}_n(z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n}), \text{ and} \\ &h^n(\ell) = hp(\underbrace{\ell, 0, \ldots, 0}_{n}) = \sum_{\mathbf{j} \in P_n(\ell)} \mathrm{Sym}_n(z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n}) = \sum_{\substack{\sum j_k = \ell \\ j_k \ge 0}} z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n} \end{aligned}$$

Note that the $hp(\mathbf{i})$ form a basis for the $\mathbb{Q}$-algebra of symmetric polynomials. However, our current interest is in the polynomials $h^n(\ell)$ which have a generating series with a particularly simple rational expression.

Proposition 3.2. *The generating series associated to the $h^r(\ell)$ satisfies*

$$\sum_{\ell \ge 0} h^r(\ell) u^\ell = [(1 - uz_1) \cdots (1 - uz_r)]^{-1}$$

Proof. This is essentially obvious:

$$\begin{aligned} [(1 - uz_1) \cdots (1 - uz_r)]^{-1} &= \left(\sum_{a_1 \ge 0} (uz_1)^{a_1} \right) \cdots \left(\sum_{a_r \ge 0} (uz_r)^{a_r} \right) \\ &= \sum_{\ell \ge 0} u^\ell \cdot \Big[\sum_{\substack{\sum a_i = \ell \\ a_i \ge 0}} z_1^{a_1} \cdots z_r^{a_r} \Big] \end{aligned}$$

It is clear from the definitions above that the coefficient of u^ℓ in the given expression is $h^r(\ell)$. ☐

Let $x_1, x_2, \ldots, x_n$ be indeterminates. We now define our family of Hecke operators as symmetric polynomials in $\mathbb{Q}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n}$: For $1 \le k \le n$, define

$$t_k^n(p^\ell) = h^{\binom{n}{k}}(\ell) \Big|_{\substack{z_i \mapsto \sigma_i(x_1 x_2 \cdots x_k) \\ \sigma_i \in S_n / \mathrm{Stab}(x_1 x_2 \cdots x_k)}}$$

This definition is not as mysterious as it first appears. We start with $h^r(\ell)$ where $r = \binom{n}{k}$ is the size of the orbit, and substitute for the variables z_i the elements in the orbit. Thus there is one element $\sigma_i \in S_n / \mathrm{Stab}(x_1 x_2 \cdots x_k)$ to correspond to each of the variables $z_1, \ldots, z_{\binom{n}{k}}$.

The above definition is sufficiently complex to warrant an example, however we note before giving detail that $t_k^n(p)$ is nothing more than the k^{th} elementary symmetric polynomial in the variables $x_1, x_2, \ldots, x_n$.

Example 3.3. *We begin by computing*

$$h^r(1) = hp(\underbrace{1,0,\ldots,0}_{r}) = \mathrm{Sym}_r(z_1) = z_1 + z_2 + \cdots + z_r.$$

Then for example,

$$t_1^n(p) = h^n(1)\Big|_{\substack{z_i \mapsto \sigma_i(x_1) \\ \sigma_i \in S_n/\,\mathrm{Stab}(x_1)}} = x_1 + x_2 + \cdots + x_n = s_1(x_1, x_2, \ldots, x_n),$$

the first elementary symmetric polynomial, and in general

$$t_k^n(p) = h^{\binom{n}{k}}(1)\Big|_{\substack{z_i \mapsto \sigma_i(x_1x_2\cdots x_k) \\ \sigma_i \in S_n/\,\mathrm{Stab}(x_1x_2\cdots x_k)}} = \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k} = s_k(x_1, x_2, \ldots, x_n),$$

the k^{th} elementary symmetric polynomial.

Remark 3.4. *First recall that under the Satake isomorphism, the full local Hecke algebra $\mathcal{H}_p$ is isomorphic to $\mathbb{Q}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n} = \mathbb{Q}[x_1, \ldots, x_n]^{S_n}[(x_1 \cdots x_n)^{-1}]$, while the integral local Hecke algebra $\underline{\mathcal{H}}_p \cong \mathbb{Q}[x_1, \ldots, x_n]^{S_n}$. Then note that the polynomials $t_k^n(p^\ell)$ are symmetric polynomials in the variables $x_1, x_2, \ldots, x_n$, so our operators $t_k^n(p^\ell)$ lie in the integral local Hecke algebra.*

Turning to generating series, we find

Proposition 3.5. *For $k = 1, 2, \ldots, n$, the generating series for the Hecke operators $t_k^n(p^\ell)$ is a rational function:*

$$\sum_{\ell \ge 0} t_k^n(p^\ell) u^\ell = \Big[\prod_{\sigma \in S_n/\,\mathrm{Stab}(x_1x_2\cdots x_k)} (1 - u\sigma(x_1x_2\cdots x_k)) \Big]^{-1}$$

Proof. From Proposition 3.2, we know that

$$\sum_{\ell \ge 0} h^{\binom{n}{k}}(\ell) u^\ell = \Big[\prod_{i=1}^{\binom{n}{k}} (1 - uz_i) \Big]^{-1}.$$

But

$$t_k^n(p^\ell) = h^{\binom{n}{k}}(\ell)\Big|_{\substack{z_i \mapsto \sigma_i(x_1x_2\cdots x_k) \\ \sigma_i \in S_n/\,\mathrm{Stab}(x_1x_2\cdots x_k)}}$$

which completes the proof. ☐

Example 3.6. *Consider the case of $n = 4$. We have the four Hecke series:*

$$\sum_{\ell \geq 0} t_1^4(p^\ell)u^\ell = [(1-ux_1)(1-ux_2)(1-ux_3)(1-ux_4)]^{-1}$$
$$\sum_{\ell \geq 0} t_2^4(p^\ell)u^\ell = [(1-ux_1x_2)(1-ux_1x_3)(1-ux_1x_4)(1-ux_2x_3)(1-ux_2x_4)(1-ux_3x_4)]^{-1}$$
$$\sum_{\ell \geq 0} t_3^4(p^\ell)u^\ell = [(1-ux_1x_2x_3)(1-ux_1x_2x_4)(1-ux_1x_3x_4)(1-ux_2x_3x_4)]^{-1}$$
$$\sum_{\ell \geq 0} t_4^4(p^\ell)u^\ell = [(1-ux_1x_2x_3x_4)]^{-1}$$

Remark 3.7. *Later in this paper we establish a connection between these t_k^n and classical Hecke operators. Accepting such a connection for now, note that the Euler factor corresponding to t_k^n has degree $\binom{n}{k}$ and the monomials are the kth exterior powers of the x_i's. Then, observing that the Euler factor corresponding to $\sum t_1^n(p^\ell)u^\ell$ seems most basic to GL_n and that the Euler factor corresponding to $\sum t_k^n(p^\ell)u^\ell$ has degree $\binom{n}{k}$ perhaps suggests a correspondence between forms on GL_n and forms on $GL_{\binom{n}{k}}$ via the exterior power L-series.*

4 The Spherical Map

Now that we have defined interesting Hecke operators in the polynomial ring, we need to invert the spherical map to characterize their "classical" counterparts.

Let $\Gamma = GL_n(\mathbb{Z})$ and $G = GL_n(\mathbb{Z}[p^{-1}])$. Andrianov [2] defines the spherical map $\omega : \Gamma\backslash G \to \mathbb{Q}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, by observing that every right coset has a unique representative in upper-triangular form with powers of p on the diagonal, and entries in any given column taken modulo the corresponding diagonal entry (analogous to Hermite normal form). On such representatives he defines the map (extending linearly) by:

$$\omega\left(\Gamma\begin{pmatrix} p^{b_1} & * & * \\ \vdots & \ddots & * \\ 0 & \ldots & p^{b_n}\end{pmatrix}\right) = p^{-m}x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}$$

where $\sum ib_i = m$ and the b_i are integers. He later proves that the restriction of ω to H_p (viewing the double coset as a sum of right cosets) gives an isomorphism onto the ring of symmetric polynomials $\mathbb{Q}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n}$.

4.1 Inverting the spherical map

As we mentioned earlier, we may assume we are working with the integral Hecke ring $\underline{H}_p$, so we need to analyze the image of the spherical map on double cosets $\Gamma\xi\Gamma$ with $\xi \in GL_n(\mathbb{Z}[p^{-1}]) \cap M_n(\mathbb{Z})$, $\det(\xi) = p^n$. Restricted to such double cosets, the spherical map has image equal to the space spanned by symmetric polynomials of total degree n. In fact, with respect to suitable bases, the restriction of the spherical map to the space of cosets spanned by double cosets $\Gamma\xi\Gamma$ with $\det(\xi) = p^n$ has a matrix which is upper triangular. Given this fact, the spherical map is (at least in principle) easy to invert. That the spherical map has this property could no doubt be deduced from the analogous result regarding representations of p-adic groups ([5],[3]), but our method of proof is also computationally valuable, so we include it here, especially since we need to find some explicit inverse images.

As noted above, for positive integers m and n, $P_n(m)$ is the set of partitions of m into n pieces with the lexicographic ordering. Note that in particular, the elements of $P_n(m)$ are linearly ordered.

By standard elementary divisor theory, for any $g \in GL_n(\mathbb{Z}[p^{-1}]) \cap M_n(\mathbb{Z})$, with $\det g = p^m$, $\Gamma g\Gamma = \Gamma \operatorname{diag}(p^{b_1}, p^{b_2}, \dots, p^{b_n})\Gamma$ for precisely one $\mathbf{b} = (b_1, b_2, \dots, b_n) \in P_n(m)$. For simplicity, write $\Gamma p^{\mathbf{b}}\Gamma$ for $\Gamma \operatorname{diag}(p^{b_1}, p^{b_2}, \dots, p^{b_n})\Gamma$, and write $\operatorname{Sym}_n(x^{\mathbf{b}})$ for $\operatorname{Sym}_n(x_1^{b_1} \cdots x_n^{b_n})$, the symmetrized polynomial under the action of the symmetric group.

Now fix the integer m, and list the elements of $P_n(m)$ in order: $P_n(m) = \{\mathbf{a}_1 < \mathbf{a}_2 < \cdots < \mathbf{a}_t\}$. Consider the double cosets $\{\Gamma p^{\mathbf{a}_i}\Gamma\}$ and the symmetric polynomials $\{\operatorname{Sym}_n(x^{\mathbf{a}_i})\}$ with an ordering induced from the one on $P_n(m)$. Each set is a linearly independent subset of the appropriate vector space. Consider the restriction of Andrianov's spherical map $\omega : H_p \to \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, to the subspace generated by the $\{\Gamma p^{\mathbf{a}_i}\Gamma\}$. The image of the spherical map restricted to this domain is spanned by the $\{\operatorname{Sym}_n(x^{\mathbf{a}_i})\}$, and we show below that the matrix of this linear map is upper triangular. We begin with a proposition of use in its own right.

Proposition 4.1. *Let $n \geq 2$ and $m \geq 1$ be integers, and let $\mathbf{b} \in P_n(m)$. Then*

$$\Gamma p^{\mathbf{b}}\Gamma = \bigcup \Gamma \begin{pmatrix} p^{a_1} & * & * \\ \vdots & \ddots & * \\ 0 & \dots & p^{a_n} \end{pmatrix} \quad \textit{(disjoint)}$$

where, in any given column, the entries are taken modulo the corresponding diagonal entry. Moreover, for some $\sigma \in S_n$, $\mathbf{a} = (a_{\sigma(1)}, \dots, a_{\sigma(n)}) \in P_n(m)$ and $\mathbf{a} \leq \mathbf{b}$.

Proof. All but the last statement is established in Lemma 2.7 of [2]. Since all diagonal entries are non-negative powers of p and any right coset representative must have the same determinant as $p^{\mathbf{b}}$, it follows that there is a $\sigma \in S_n$ with $\mathbf{a} = (a_{\sigma(1)}, \dots, a_{\sigma(n)}) \in P_n(m)$. For any right coset occurring in the decomposition of the double coset $\Gamma p^{\mathbf{b}}\Gamma$, we must have that the corresponding double cosets agree, that is $\Gamma \begin{pmatrix} p^{a_1} & * & * \\ \vdots & \ddots & * \\ 0 & \dots & p^{a_n} \end{pmatrix} \Gamma = \Gamma p^{\mathbf{b}}\Gamma$. From Theorem II.10 of [4], this is true if and only if the two matrices, $\begin{pmatrix} p^{a_1} & * & * \\ \vdots & \ddots & * \\ 0 & \dots & p^{a_n} \end{pmatrix}$ and $\operatorname{diag}(p^{b_1}, p^{b_2}, \dots, p^{b_n})$, have the same determinantal divisors. Recall that the kth determinantal divisor is simply the gcd of the determinants of all $k \times k$ sub-matrices of the matrix, and that the quotients of successive determinantal divisors yield the elementary divisors. By a $k \times k$ sub-matrix we mean the matrix obtained by fixing k row and k column indices. To proceed we need a small

Lemma 4.2. *Let* $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n) \in P_n(m)$ *with* $\mathbf{b} < \mathbf{a}$. *There is a smallest index* k *for which* $b_k > a_k$. *For this* k, $\sum_k^n b_i > \sum_k^n a_i$.

Proof. Since $\mathbf{b} < \mathbf{a}$ there is an index $i_0 \geq 1$ such that $b_i = a_i$ for $i < i_0$, and $b_{i_0} < a_{i_0}$. Thus $\sum_1^{i_0} b_i < \sum_1^{i_0} a_i$. But since $\mathbf{a}, \mathbf{b} \in P_n(m)$, $\sum_1^n a_i = \sum_1^n b_i$, so there must exist a smallest index k (clearly $> i_0$) for which $b_k > a_k$. It follows that $\sum_1^{k-1} b_i < \sum_1^{k-1} a_i$, so that $\sum_k^n b_i > \sum_k^n a_i$, as required. ☐

To complete the proof of the proposition, we have that $\Gamma \begin{pmatrix} p^{a_1} & * & * \\ \vdots & \ddots & * \\ 0 & \ldots & p^{a_n} \end{pmatrix}$ is a right coset representative of $\Gamma p^{\mathbf{b}} \Gamma$, and that $\mathbf{a} = (a_{\sigma(1)}, \ldots, a_{\sigma(n)}) \in P_n(m)$ for some $\sigma \in S_n$. Suppose to the contrary that $\mathbf{a} > \mathbf{b}$. Recall we have $m \geq b_1 \geq \cdots \geq b_n \geq 0$ and $m \geq a_{\sigma(1)} \geq \cdots \geq a_{\sigma(n)} \geq 0$. Denote by $d_k^{\mathbf{b}}$ and $d_k^{\mathbf{a}}$ the kth determinantal divisor of $\operatorname{diag}(p^{b_1}, \ldots, p^{b_n})$ and $\begin{pmatrix} p^{a_1} & * & * \\ \vdots & \ddots & * \\ 0 & \ldots & p^{a_n} \end{pmatrix}$ respectively.

The determinantal divisors of $\operatorname{diag}(p^{b_1}, \ldots, p^{b_n})$ are clear: $d_1^{\mathbf{b}} = p^{b_n}$, $d_2^{\mathbf{b}} = p^{b_n + b_{n-1}}$, $\ldots$, $d_n^{\mathbf{b}} = p^{b_n + \cdots + b_1}$. Since we are assuming that $\mathbf{a} > \mathbf{b}$, we have by the lemma a smallest index k for which $b_k > a_{\sigma(k)}$, and hence for which $d_{n+1-k}^{\mathbf{b}} = p^{\sum_k^n b_i} > p^{\sum_k^n a_{\sigma(i)}}$. However $d_{n+1-k}^{\mathbf{a}}$ is the gcd of all possible determinants of $(n+1-k) \times (n+1-k)$ sub-matrices, and choosing rows and columns $\sigma(k), \ldots, \sigma(n)$ for one such matrix produces a determinant smaller than $d_{n+1-k}^{\mathbf{b}}$, thus showing that $d_{n+1-k}^{\mathbf{a}} < d_{n+1-k}^{\mathbf{b}}$, a contradiction. ☐

We now have a characterization of the spherical map.

Theorem 4.3. *The matrix of the spherical map, restricted as above, is nonsingular and upper triangular. In particular, if* $\mathbf{b} \in P_n(m)$, *then*

$$\omega(\Gamma p^{\mathbf{b}} \Gamma) = \sum_{\mathbf{a} \in P_n(m),\ \mathbf{a} \leq \mathbf{b}} c(\mathbf{a}) \operatorname{Sym}_n(x^{\mathbf{a}})$$

for constants $c(\mathbf{a})$ *with* $c(\mathbf{b}) \neq 0$.

Proof. It is enough to consider ω restricted to the integral subring $\underline{H}_p^n$. We compute the image of ω on the double coset $\Gamma\operatorname{diag}(p^{b_1},\dots,p^{b_n})\Gamma$ for $\mathbf{b}=(b_1,\dots,b_n)\in P_n(m)$. From Proposition 4.1 above, the double coset is a union of right cosets $\Gamma\begin{pmatrix} p^{a_{\sigma(1)}} & * & * \\ \vdots & \ddots & * \\ 0 & \dots & p^{a_{\sigma(n)}} \end{pmatrix}$ where $\mathbf{a}=(a_1,\dots,a_n)\in P_n(m)$, $\mathbf{a}\le\mathbf{b}$, and $\sigma\in S_n$. By definition of the spherical map [2],

$$\omega\left(\Gamma\begin{pmatrix} p^{a_{\sigma(1)}} & * & * \\ \vdots & \ddots & * \\ 0 & \dots & p^{a_{\sigma(n)}} \end{pmatrix}\right) = p^{-\sum i a_{\sigma(i)}} x_1^{a_{\sigma(1)}}\cdots x_n^{a_{\sigma(n)}},$$

so to understand the image of a double coset, we obviously need to count the number of such right cosets occur in $\Gamma p^{\mathbf{b}}\Gamma$. Denote this number by $\psi^\sigma_{\mathbf{a},\mathbf{b}}$.

Then

$$\omega(\Gamma p^{\mathbf{b}}\Gamma) = \sum_{\mathbf{a}\le\mathbf{b},\ \mathbf{a}\in P_n(m)}\ \sum_{\sigma^{-1}\in S_n/\operatorname{Stab}(x^{\mathbf{a}})} \psi^\sigma_{\mathbf{a},\mathbf{b}}\, p^{-\sum i a_{\sigma(i)}}\, x_1^{a_{\sigma(1)}}\cdots x_n^{a_{\sigma(n)}}$$

Note that the inner sum is over σ^{-1} and not σ since if $\sigma(i)=j$, then $x_i^{a_{\sigma(i)}} = x_{\sigma^{-1}(j)}^{a_j}$. Moreover, the sum over σ^{-1} is not generally the same as the sum over σ unless $\operatorname{Stab}(x^{\mathbf{a}})$ is a normal subgroup of S_n.

On the other hand, $\omega(\Gamma p^{\mathbf{b}}\Gamma)$ is a symmetric polynomial, so that $\psi^\sigma_{\mathbf{a},\mathbf{b}}\, p^{-\sum i a_{\sigma(i)}} = \psi^\tau_{\mathbf{a},\mathbf{b}}\, p^{-\sum i a_{\tau(i)}}$ for any two $\sigma,\tau\in S_n$. In particular, $\psi^\sigma_{\mathbf{a},\mathbf{b}} = p^{\sum i(a_{\sigma(i)}-a_i)}\psi^1_{\mathbf{a},\mathbf{b}}$, where 1 denotes the identity permutation, and consequently

$$\omega(\Gamma p^{\mathbf{b}}\Gamma) = \sum_{\mathbf{a}\le\mathbf{b},\ \mathbf{a}\in P_n(m)} \psi^1_{\mathbf{a},\mathbf{b}}\, p^{-\sum i a_i}\ \operatorname{Sym}_n(x^{\mathbf{a}}). \tag{4.1}$$

Thus $c(\mathbf{a}) = \psi^1_{\mathbf{a},\mathbf{b}}\, p^{-\sum i a_i}$, and it is clear that $c(\mathbf{b})\ne 0$ since one of the right coset representatives of the double coset is the diagonal matrix $\operatorname{diag}(p^{b_1},\dots,p^{b_n})$. □

Remark 4.4. *Of course the coefficients $c(\mathbf{a}) = \psi^\sigma_{\mathbf{a},\mathbf{b}}\, p^{-\sum i a_{\sigma(i)}}$ for any $\sigma\in S_n$, however $\psi^1_{\mathbf{a},\mathbf{b}}$ is the easiest to compute since we are assuming the right coset has the form $\begin{pmatrix} p^{a_1} & * & * \\ \vdots & \ddots & * \\ 0 & \dots & p^{a_n} \end{pmatrix}$ with $a_1\ge a_2\ge\cdots\ge a_n$, and where the entries in the ith column can be taken modulo p^{a_i}, and hence this case presents the fewest right cosets to analyze.*

In any real computation, this simplification represents an enormous saving of time, and to some extent complexity since determining which right cosets belong to a given double coset involves computing determinantal divisors. We note that the complexity of the problem depends not so much on n or m, but on the cardinality of $P_n(m)$ which grows extremely fast with m even for small n.

Corollary 4.5. *The matrix of the inverse of the spherical map, restricted as above, is upper triangular. In particular, if $\mathbf{b} \in P_n(m)$, then*

$$\mathrm{Sym}(x^{\mathbf{b}}) = \sum_{\mathbf{a} \in P_n(m),\ \mathbf{a} \leq \mathbf{b}} d(\mathbf{a}) \omega(\Gamma p^{\mathbf{a}} \Gamma)$$

for constants $d(\mathbf{a})$.

Remark 4.6. *We point out three further observations which can be useful for simplifying such computations. Let $P_n(m) = \{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_r\}$ and assume that $\boldsymbol{\lambda}_1 < \cdots < \boldsymbol{\lambda}_r$. If $\boldsymbol{\lambda}_i = (\lambda_{i,1}, \ldots, \lambda_{i,n})$, let $\chi(\boldsymbol{\lambda}_i, \boldsymbol{\lambda}_j)$ denote the number of right cosets of the form $\Gamma \begin{pmatrix} p^{\lambda_{i,\sigma(1)}} & * & * \\ \vdots & \ddots & * \\ 0 & \ldots & p^{\lambda_{i,\sigma(n)}} \end{pmatrix}$ which occur in the double coset $\Gamma p^{\boldsymbol{\lambda}_j} \Gamma$, as σ ranges over S_n. From the discussion above, we know $\chi(\boldsymbol{\lambda}_i, \boldsymbol{\lambda}_j) = 0$ whenever $\boldsymbol{\lambda}_i > \boldsymbol{\lambda}_j$.*

Certainly, we already know that for any $\mathbf{a}, \mathbf{b} \in P_n(m)$ ($\mathbf{a} \leq \mathbf{b}$)

$$\chi(\mathbf{a}, \mathbf{b}) = \sum_{\sigma^{-1} \in S_n / \mathrm{Stab}(x^{\mathbf{a}})} \psi^{\sigma}_{\mathbf{a},\mathbf{b}} = \sum_{\sigma^{-1} \in S_n / \mathrm{Stab}(x^{\mathbf{a}})} p^{\sum i(a_{\sigma(i)} - a_i)} \psi^1_{\mathbf{a},\mathbf{b}}, \tag{4.2}$$

but we also have the following relations:

1. *$\mu(\boldsymbol{\lambda}_j) = \chi(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_j) + \cdots + \chi(\boldsymbol{\lambda}_j, \boldsymbol{\lambda}_j)$ is the degree (i.e., the total number of right cosets in the double coset) of the double coset $\Gamma p^{\boldsymbol{\lambda}_j} \Gamma$ which is explicitly computed in Lemma 6 of [1].*

2. *$\chi(\boldsymbol{\lambda}_j, \boldsymbol{\lambda}_j) + \cdots + \chi(\boldsymbol{\lambda}_j, \boldsymbol{\lambda}_r)$ is the total number of possible right cosets with $p^{\lambda_{j,k}}$'s (j fixed) on the diagonal. This is trivial to compute since for any given upper triangular matrix, all the entries above are taken modulo the diagonal entry in that column.*

3. *Refining the previous item, we have that*

$$\psi^1_{\boldsymbol{\lambda}_j, \boldsymbol{\lambda}_j} + \cdots + \psi^1_{\boldsymbol{\lambda}_j, \boldsymbol{\lambda}_r} = p^{\ell}, \quad \ell = \sum_{k=1}^{r} \lambda_{j,k}(k-1)$$

The first two observations produce $2r$ linear relations among the $r(r+1)/2$ variables $\chi(\boldsymbol{\lambda}_i, \boldsymbol{\lambda}_j)$. These together with the third observation will be used in the example below to further reduce the computational load of inverting the spherical map.

4.2 Some Examples

In this section we provide two examples one trivial, one not, which will be used to define our new families of Hecke operators for GL_4, and which demonstrate the observations given in the previous section. We shall compute the inverse image (under the spherical map) of $t^4_1(p^2)$ and $t^4_2(p^2)$.

We consider first the almost trivial case of $t_1^4(p^2)$ which requires we analyze matrices of determinant p^2. $P_4(2) = \{\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2\}$ where $\boldsymbol{\lambda}_1 = (1,1,0,0) < \boldsymbol{\lambda}_2 = (2,0,0,0)$. To ease the notation, we write ψ_{ij} for $\psi^1_{\boldsymbol{\lambda}_i,\boldsymbol{\lambda}_j}$ in both examples below, and denote the kth determinantal divisor $d_k^{\boldsymbol{\lambda}_i}$ as d_k^i.

There is only one right coset with diagonal $(p^2,1,1,1)$ since all the upper triangular entries are read modulo the diagonal entry, so it is clear that $\psi_{2,2} = 1$. The right cosets of the form $\Gamma \begin{pmatrix} p & a & 0 & 0 \\ & p & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}$ belong either to the double coset $\Gamma \operatorname{diag}(p,p,1,1)\Gamma$ or to $\Gamma \operatorname{diag}(p^2,1,1,1)\Gamma$, and the third determinantal divisor easily determines which for us, with $d_3^1 = p$ while $d_3^2 = 1$. For a typical coset we see that the third determinantal divisor is $d_3 = \gcd(p,a)$. Now a runs mod p, so we immediately see that $\psi_{11} = 1$ and $\psi_{12} = p-1$.

Recall from equation 4.1, $\omega(\Gamma p^{\mathbf{b}}\Gamma) = \sum_{\mathbf{a}\le\mathbf{b},\ \mathbf{a}\in P_n(m)} \psi^1_{\mathbf{a},\mathbf{b}}\, p^{-\sum i a_i} \operatorname{Sym}(x^{\mathbf{a}})$. Let $A = (a_{ij})$ denote the (upper triangular) matrix of the (restricted) spherical map, so $\omega(\Gamma p^{\boldsymbol{\lambda}_j}\Gamma) = \sum_{i\le j} a_{ij} \operatorname{Sym}(x^{\boldsymbol{\lambda}_i})$. If $\boldsymbol{\lambda}_i = (\lambda_{i,1},\ldots,\lambda_{i,n})$, then $a_{ij} = p^{-\sum k\lambda_{i,k}}\psi_{ij}$, thus $A = \begin{pmatrix} p^{-3} & p^{-3}(p-1) \\ 0 & p^{-2} \end{pmatrix}$.

It's inverse is $A^{-1} = \begin{pmatrix} p^3 & -p^2(p-1) \\ 0 & p^2 \end{pmatrix}$, that is

$$\begin{aligned} \omega^{-1}(\operatorname{Sym}_4(x^{\boldsymbol{\lambda}_1})) &= \omega^{-1}(\operatorname{Sym}_4(x_1x_2)) = p^3\Gamma \operatorname{diag}(p,p,1,1)\Gamma \text{ and} \\ \omega^{-1}(\operatorname{Sym}_4(x^{\boldsymbol{\lambda}_2})) &= \omega^{-1}(\operatorname{Sym}_4(x_1^2)) = p^2\Gamma \operatorname{diag}(p^2,1,1,1)\Gamma - p^2(p-1)\Gamma \operatorname{diag}(p,p,1,1)\Gamma \end{aligned}$$

Since $t_1^4(p^2) = \operatorname{Sym}_4(x_1^2) + \operatorname{Sym}_4(x_1x_2)$, it follows that

$$\omega^{-1}(t_1^4(p^2)) = p^2\Gamma \operatorname{diag}(p^2,1,1,1)\Gamma + p^2\Gamma \operatorname{diag}(p,p,1,1)\Gamma.$$

Next we consider the harder example of $t_2^4(p^2)$. This requires that we study matrices of determinant p^4. $P_4(4) = \{\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_5\}$, where $\boldsymbol{\lambda}_1 = (1,1,1,1) < \boldsymbol{\lambda}_2 = (2,1,1,0) < \boldsymbol{\lambda}_3 = (2,2,0,0) < \boldsymbol{\lambda}_4 = (3,1,0,0) < \boldsymbol{\lambda}_5 = (4,0,0,0)$. We need to compute ψ_{ij} for $1 \le i \le j \le 5$. For $2 \le i \le 5$, the ψ_{ij} are computed by means of analyzing determinantal divisors. For example for $i = 2$, the task is to determine how many right cosets $\Gamma \begin{pmatrix} p^2 & a & b & 0 \\ & p & c & 0 \\ & & p & 0 \\ & & & 1 \end{pmatrix}$ (with $a,b,c \pmod p$) belong to the double cosets $\Gamma \operatorname{diag}(p^2,p,p,1)\Gamma$, $\Gamma \operatorname{diag}(p^2,p^2,1,1)\Gamma$, $\Gamma \operatorname{diag}(p^3,p,1,1)\Gamma$, and $\Gamma \operatorname{diag}(p^4,1,1,1)\Gamma$. These numbers will be (respectively) $\psi_{22},\psi_{23},\psi_{24},\psi_{25}$. If we put $\Psi = (\psi_{ij})$, then careful counting with determinantal divisors gives us all of Ψ except the first row:

$$\Psi = \begin{pmatrix} * & * & * & * & * \\ & 1 & p-1 & 2p(p-1) & p(p-1)^2 \\ & & 1 & p-1 & p(p-1) \\ & & & 1 & p-1 \\ & & & & 1 \end{pmatrix}$$

Note that as a check, the row sums are consistent with observation 3 in Remark 4.6. In principle, the first row could also be computed in this fashion although a quick inspection of the problem reveals a myriad of cases which must be analyzed to determine the determinantal divisors. So instead, we take advantage of the other observations we made in the previous section.

First we observe by equation 4.2, that $\chi_{1,j} = \psi_{1j}$ since the stabilizer of $x^{\boldsymbol{\lambda}_1} = x_1x_2x_3x_4$ is all of S_4. Then we note from Remark 4.6 that $\chi_{1,j} = \mu(\boldsymbol{\lambda}_j) - \chi_{2,j} - \chi_{3,j} - \cdots - \chi_{j,j}$, where the $\mu(\boldsymbol{\lambda}_j)$ is the degree of the double coset $\Gamma p^{\boldsymbol{\lambda}_j}\Gamma$ which is explicitly computed in Lemma 6 of [1]. Unfortunately, we must also compute the $\chi_{i,j}$ $(i > 1)$ from our knowledge of the ψ_{ij} using equation 4.2. If we define $\nu(\boldsymbol{\lambda}_i)$ by $\chi_{i,j} = \nu(\boldsymbol{\lambda}_i)\psi_{ij}$, we compute that

	$\mu(\boldsymbol{\lambda}_i)$	$\nu(\boldsymbol{\lambda}_i)$
$\boldsymbol{\lambda}_2$	$p(p^2+p+1)(p^2+1)(p+1)$	$1+2p+3p^2+3p^4+2p^5+p^6$
$\boldsymbol{\lambda}_3$	$p^4(p^2+p+1)(p^2+1)$	$1+p^2+2p^4+p^6+p^8$
$\boldsymbol{\lambda}_4$	$p^5(p^2+p+1)(p^2+1)(p+1)$	$1+p+2p^2+p^4+2p^5+ p^6+2p^8+p^9+p^{10}$
$\boldsymbol{\lambda}_5$	$p^9(p^2+1)(p+1)$	$1+p^4+p^8+p^{12}$

Putting this together we conclude:

$$\begin{aligned}
\psi_{11} &= \chi_{11} = 1\\
\psi_{12} &= \chi_{12} = 3p^3 - p^2 - p - 1\\
\psi_{13} &= \chi_{13} = 2p^4 - 3p^3 + p\\
\psi_{14} &= \chi_{14} = 3p^5 - 5p^4 + p^3 + p^2\\
\psi_{15} &= \chi_{15} = p^6 - 3p^5 + 3p^4 - p^3
\end{aligned}$$

Now if $A = (a_{ij})$ once again represents the matrix of the spherical map with respect to the canonical bases, then $a_{ij} = p^{-\sum k\lambda_{i,k}} \cdot \psi_{ij}$, and we have

$$A = \begin{pmatrix}
\frac{1}{p^{10}} & \frac{3\,p^3 - p^2 - p - 1}{p^{10}} & \frac{2\,p^4 - 3\,p^3 + p}{p^{10}} & \frac{3\,p^5 - 5\,p^4 + p^3 + p^2}{p^{10}} & \frac{p^6 - 3\,p^5 + 3\,p^4 - p^3}{p^{10}}\\
0 & \frac{1}{p^7} & \frac{p-1}{p^7} & \frac{2p(p-1)}{p^7} & \frac{p(p-1)^2}{p^7}\\
0 & 0 & \frac{1}{p^6} & \frac{p-1}{p^6} & \frac{p(p-1)}{p^6}\\
0 & 0 & 0 & \frac{1}{p^5} & \frac{p-1}{p^5}\\
0 & 0 & 0 & 0 & \frac{1}{p^4}
\end{pmatrix}$$

It's inverse A^{-1} is:

$$\begin{pmatrix} p^{10} & -p^7(3p^3-p^2-p-1) & p^6(p^4-p^3-p+1) & p^5(2p^5-p^4-2p^3+1) & -p^4(p^6-p^5-p^4+p^2+p-1) \\ 0 & p^7 & -p^6(p-1) & -p^5(p-1)(p+1) & p^4(p-1)^2(p+1) \\ 0 & 0 & p^6 & -p^5(p-1) & -p^4(p-1) \\ 0 & 0 & 0 & p^5 & -p^4(p-1) \\ 0 & 0 & 0 & 0 & p^4 \end{pmatrix}$$

We compute that $t_2^4(p^2) = \mathrm{Sym}_4(x^{\boldsymbol{\lambda}_3}) + \mathrm{Sym}_4(x^{\boldsymbol{\lambda}_2}) + 3\,\mathrm{Sym}_4(x^{\boldsymbol{\lambda}_1})$, so

$$\begin{aligned}\omega^{-1}(t_2^4(p^2)) = {} & [(p^4-p^3+1-p)p^6\Gamma p^{\boldsymbol{\lambda}_1}\Gamma - p^6(p-1)\Gamma p^{\boldsymbol{\lambda}_2}\Gamma + p^6\Gamma p^{\boldsymbol{\lambda}_3}\Gamma] \\ & + [-p^7(3p^3-p^2-p-1)\Gamma p^{\boldsymbol{\lambda}_1}\Gamma + p^7\Gamma p^{\boldsymbol{\lambda}_2}\Gamma] + 3[p^{10}\Gamma p^{\boldsymbol{\lambda}_1}\Gamma] \\ = {} & p^6\left[(p^4+p^2+1)\Gamma p^{\boldsymbol{\lambda}_1}\Gamma + \Gamma p^{\boldsymbol{\lambda}_2}\Gamma + \Gamma p^{\boldsymbol{\lambda}_3}\Gamma\right] \\ = {} & p^6\left[(p^4+p^2+1)\Gamma\,\mathrm{diag}(p,p,p,p)\Gamma + \Gamma\,\mathrm{diag}(p^2,p,p,1)\Gamma + \Gamma\,\mathrm{diag}(p^2,p^2,1,1)\Gamma\right]\end{aligned}$$

5 New Families of Hecke Operators

Recall that we have defined families of polynomial Hecke operators

$$t_k^n(p^\ell) = h^{\binom{n}{k}}(\ell)\Big|_{\substack{z_i\mapsto\sigma_i(x_1x_2\cdots x_k)\\ \sigma_i\in S_n/\operatorname{Stab}(x_1x_2\cdots x_k)}}$$

where $h^r(\ell) = \sum_{\mathbf{j}\in P_r(\ell)} \mathrm{Sym}_r(z_1^{j_1}z_2^{j_2}\cdots z_r^{j_r}) = \sum_{\substack{\sum j_k=\ell\\ j_k\ge 0}} z_1^{j_1}z_2^{j_2}\cdots z_r^{j_r}$, and which satisfy

$$\sum_{\ell\ge 0} t_k^n(p^\ell)u^\ell = \Big[\prod_{\sigma\in S_n/\operatorname{Stab}(x_1x_2\cdots x_k)} (1-u\sigma(x_1x_2\cdots x_k))\Big]^{-1}, \tag{5.1}$$

and $t_k^n(p)$ is the kth elementary symmetric polynomial in the variables $x_1,\dots,x_n$.

One could attempt simply to apply the inverse of the spherical map to the elements $t_k^n(p^\ell)$, but this is neither practical, nor particularly illuminating. Instead, we realize that even for GL_2 the Hecke operators are most clearly defined by defining some base cases and the recursion relations satisfied by the operators, so this is how we proceed here.

Equation 5.1 implicitly contains the recursion relations for the $t_k^n(p^\ell)$ via

$$\Big[\sum_{\ell\ge 0} t_k^n(p^\ell)u^\ell\Big]\cdot\Big[\prod_{\sigma\in S_n/\operatorname{Stab}(x_1x_2\cdots x_k)} (1-u\sigma(x_1x_2\cdots x_k))\Big] = 1 \tag{5.2}$$

In the following sections, we completely characterize $t_1^n(p^\ell)$ and $t_n^n(p^\ell)$ for any n. Then as we saw hinted at in Example 3.6, we investigate a sort of duality between t_k^n and t_{n-k}^n. The remaining operators are difficult to characterize in general, but in the last section, we work out the details for GL_4. This is the first really interesting case as it associates an "Euler factor" of degree $> n$ to a Hecke operator for GL_n.

5.1 The operators $t_1^n(p^\ell)$ and $t_n^n(p^\ell)$

The case $k = 1$ recovers the classical Hecke operators (see §3.2 of [6]). In particular, we see that if $s_k = s_k(x_1, \ldots, x_n) = t_k^n(p)$ is the kth elementary symmetric polynomial, then

$$\left[\sum_{\ell \geq 0} t_1^n(p^\ell)u^\ell\right] \cdot \left[1 - s_1 u + s_2 u^2 - s_3 u^3 + \cdots + (-1)^n s_n u^n\right] = 1,$$

or (letting $t_1^n(p^j) = 0$ for $j < 0$),

$$\begin{aligned} t_1^n(p^\ell) &= t_1^n(p^{\ell-1})s_1 - t_1(p^{\ell-2})s_2 + \cdots + (-1)^{n-1} t_1^n(p^{\ell-n})s_n \\ &= t_1^n(p^{\ell-1})t_1^n(p) - t_1(p^{\ell-2})t_2^n(p) + \cdots + (-1)^{n-1} t_1^n(p^{\ell-n})t_n^n(p) \end{aligned}$$

At the other extreme are the operators t_n^n which satisfy $\sum_{\ell \geq 0} t_n^n(p^\ell)u^\ell = [1 - ux_1 \cdots x_n]^{-1}$ which is the sum of a geometric series, yielding

$$t_n^n(p^\ell) = t_n^n(p)^\ell = [s_n(x_1, \ldots, x_n)]^\ell$$

5.2 Relations between t_k^n and t_{n-k}^n

We note that there is a sort of duality between $t_k^n(p^\ell)$ and $t_{n-k}^n(p^\ell)$. Since $\binom{n}{k} = \binom{n}{n-k}$, we see that $t_{n-k}^n(p^\ell) = h^{\binom{n}{n-k}}\Big|_{\substack{z_i \mapsto \sigma_i(x_1 x_2 \cdots x_{n-k}) \\ \sigma_i \in S_n / \operatorname{Stab}(x_1 x_2 \cdots x_{n-k})}} = h^{\binom{n}{k}}\Big|_{\substack{z_i \mapsto x_1 x_2 \cdots x_n / \sigma_i(x_1 x_2 \cdots x_k) \\ \sigma_i \in S_n / \operatorname{Stab}(x_1 x_2 \cdots x_k)}}$ while $t_k^n(p^\ell) = h^{\binom{n}{k}}(\ell)\Big|_{\substack{z_i \mapsto \sigma_i(x_1 x_2 \cdots x_k) \\ \sigma_i \in S_n / \operatorname{Stab}(x_1 x_2 \cdots x_k)}}$.

The correspondence is most easily seen between t_1 and t_{n-1}. An elementary computation shows

$$s_n(x_1, \ldots, x_n) \cdot s_k(x_1^{-1}, \ldots, x_n^{-1}) = s_{n-k}(x_1, \ldots, x_n)$$

and hence (letting $s_n = s_n(x_1, \ldots, x_n) = x_1 \cdots x_n$),

$$s_k(s_n/x_1, \ldots, s_n/x_n) = s_n^{k-1}(x_1, \ldots, x_n) \cdot s_{n-k}(x_1, \ldots, x_n)$$

Thus,

$$\begin{aligned} \sum_{\ell \geq 0} t_1^n(p^\ell)u^\ell &= \left[\prod_{\sigma \in S_n / \operatorname{Stab}(x_1)} (1 - u\sigma(x_1))\right]^{-1} \\ &= \left[1 - us_1(x_1, \ldots, x_n) + u^2 s_2(x_1, \ldots, x_n) + \cdots + (-1)^n u^n s_n(x_1, \ldots, x_n)\right]^{-1} \end{aligned}$$

while for t^n_{n-1} we have

$$\begin{aligned}
\sum_{\ell\ge 0} t^n_{n-1}(p^\ell)u^\ell &= \Big[\prod_{\sigma\in S_n/\operatorname{Stab}(x_1x_2\cdots x_{n-1})} (1-u\sigma(x_1x_2\cdots x_{n-1}))\Big]^{-1}\\
&= \left[\prod_{\sigma\in S_n/\operatorname{Stab}(x_1)} (1-us_n/\sigma(x_1))\right]^{-1}\\
&= \left[\prod_{j=1}^{n}(1-us_n/x_j)\right]^{-1}\\
&= [1-us_1(s_n/x_1,\dots,s_n/x_n)+\cdots+(-1)^n u^n s_n(s_n/x_1,\dots,s_n/x_n)]^{-1}\\
&= [1-us_{n-1}(x_1,\dots,x_n)+u^2 s_n(x_1,\dots,x_n)s_{n-2}(x_1,\dots,x_n)\\
&\qquad -u^3 s_n^2(x_1,\dots,x_n)s_{n-3}(x_1,\dots,x_n)+\cdots+(-1)^n u^n s_n^{n-1}(x_1,\dots,x_n)]^{-1},
\end{aligned}$$

showing the nice symmetry.

5.3 The recursion relations for Hecke operators on GL_2, GL_3, GL_4

Having done what we can for the operators for general n, here we work out all the recursion relations defining the Hecke operators on GL_n for $n = 2, 3, 4$. For small n there is little choice in how these relations are written down, but as we shall see for $n \ge 4$, it is possible to write down multiple recursion relations which appear quite distinct. GL_4 is the first really interesting case, however for completeness we write down the recursion relations for $n = 2, 3, 4$. Throughout, we put $t^n_k(p^j) = 0$ for $j < 0$ (and $t^n_k(1) = 1$ by definition), and recall $t^n_n(p^\ell) = t^n_n(p)^\ell$.

The relations for the case $n = 2$ simply consists of a base case and one relation:

$$\begin{aligned}
&t^2_k(p) = s_k(x_1, x_2), \quad k = 1, 2\\
&t^2_1(p^\ell) = t^2_1(p^{\ell-1})t^2_1(p) - t^2_1(p^{\ell-2})t^2_2(p)
\end{aligned}$$

For $n = 3$, we have a base case and two nontrivial relations:

$$\begin{aligned}
&t^3_k(p) = s_k(x_1, x_2, x_3), \quad k = 1, 2, 3\\
&t^3_1(p^\ell) = t^3_1(p^{\ell-1})t^3_1(p) - t^3_1(p^{\ell-2})t^3_2(p) + t^3_1(p^{\ell-3})t^3_3(p)\\
&t^3_2(p^\ell) = t^3_2(p^{\ell-1})t^3_2(p) - t^3_2(p^{\ell-2})t^3_1(p)t^3_3(p) + t^3_2(p^{\ell-3})t^3_3(p)^2
\end{aligned}$$

For $n = 4$, we analogously have a base case and three nontrivial relations. The relations for $t^4_1(p^\ell)$ and $t^4_3(p^\ell)$ are fairly easy to write down from what we have noted above, but the relation for $t^4_2(p^\ell)$ is somewhat more involved, and indeed is much more suggestive of the general case $t^n_k(p^\ell)$ for $n > 4$. The relations for $t^4_2(p^\ell)$ are deduced from the identity:

$$[\sum_{\ell\ge 0} t^4_2(p^\ell)u^\ell]\cdot[(1-ux_1x_2)(1-ux_1x_3)(1-ux_1x_4)(1-ux_2x_3)(1-ux_2x_4)(1-ux_3x_4)] = 1$$

To write things in closed from, we adopt some ad hoc notation to simplify the exposition. Let $t(a_1, a_2, a_3, a_4)$ be the symmetric polynomial $\mathrm{Sym}_4(x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4})$. Then

$$\begin{aligned}&(1-ux_1x_2)(1-ux_1x_3)(1-ux_1x_4)(1-ux_2x_3)(1-ux_2x_4)(1-ux_3x_4) = \\ &\quad 1 - ut(1,1,0,0) + u^2(t(2,1,1,0) + 3t(1,1,1,1)) - u^3(t(3,1,1,1) + t(2,2,2,0) + 2t(2,2,1,1)) \\ &\quad + u^4(t(3,2,2,1) + 3t(2,2,2,2)) - u^5 t(3,3,2,2) + u^6 t(3,3,3,3)\end{aligned}$$

There are often many ways to represent the coefficients above in terms of the $t_k^4(p^\ell)$. For example, the coefficient of u^3 is

$$\begin{aligned}&t(3,1,1,1) + t(2,2,2,0) + 2t(2,2,1,1) = \\ &t_4^4(p)t_1^4(p^2) + 2t_3^4(p^2) - t_3^4(p)^2 = \\ &t_4^4(p)(t_1^4(p^2) + t_3^4(p))\end{aligned}$$

and the shape of the recursion relations we write down depends upon the choice of the representation. Here we opt for simplicity, and express

$$\begin{aligned}&(1-ux_1x_2)(1-ux_1x_3)(1-ux_1x_4)(1-ux_2x_3)(1-ux_2x_4)(1-ux_3x_4) = \\ &\quad 1 - ut_2^4(p) + u^2(t_2^4(p)^2 - t_2^4(p^2)) - u^3 t_4^4(p)(t_1(p^2) + t_3^4(p)) \\ &\quad + u^4 t_4^4(p)(t_2^4(p)^2 - t_2^4(p^2)) - u^5 t_4^4(p)^2 t_2^4(p) + u^6 t_4^4(p)^3\end{aligned}$$

From this we deduce the recursion relations for $n = 4$:

$$\begin{aligned}&t_k^4(p) = s_k(x_1, x_2, x_3, x_4), \quad k = 1,2,3,4 \\ &t_1^4(p^\ell) = t_1^4(p^{\ell-1})t_1^4(p) - t_1^4(p^{\ell-2})t_2^4(p) + t_1^4(p^{\ell-3})t_3^4(p) - t_1^4(p^{\ell-4})t_4^4(p) \\ &t_2^4(p^\ell) = t_2^4(p^{\ell-1})t_2^4(p) - t_2^4(p^{\ell-2})(t_2^4(p)^2 - t_2^4(p^2)) + t_2^4(p^{\ell-3})t_4^4(p)(t_1^4(p^2) + t_3^4(p)) \\ &\qquad - t_2^4(p^{\ell-4})t_4^4(p)(t_2^4(p)^2 - t_2^4(p^2)) + t_2^4(p^{\ell-5})t_2^4(p)t_4^4(p)^2 - t_2^4(p^{\ell-6})t_4^4(p)^3 \\ &t_3^4(p^\ell) = t_3^4(p^{\ell-1})t_3^4(p) - t_3^4(p^{\ell-2})t_2^4(p)t_4^4(p) + t_3^4(p^{\ell-3})t_1^4(p)t_4^4(p)^2 - t_3^4(p^{\ell-4})t_4(p)^3\end{aligned}$$

6 The Hecke operators in the classical setting

We finally reach the point where we can define the classical Hecke operators which correspond to our polynomial ones. These are the operators defined in terms of double cosets which should be of number-theoretic interest because of their well structured generating series.

Recall our notation for the standard generators of H_p:

$$\pi_k^n(p) = \Gamma\,\mathrm{diag}(\underbrace{p, \ldots, p}_{k}, \underbrace{1, \ldots, 1}_{n-k})\Gamma, \quad k = 1, \ldots, n$$

We let $T_k^n(p^\ell)$ be the inverse image under the spherical map of $t_k^n(p^\ell)$. Since the spherical map ω is an isomorphism, the $T_k^n(p^\ell)$ will satisfy exactly the same recursion relations as the $t_k^n(p^\ell)$. They will also have a generating series $\sum T_k^n(p^\ell)u^\ell$ which are rational functions of exactly the same form as the polynomial Hecke operators, in particular, $\sum T_k^n(p^\ell)u^\ell = [q_k^n(u)]^{-1}$ where q_k^n is a polynomial of degree $\binom{n}{k}$.

Finally, we note that to define the operators $T_k^n(p^\ell)$, we need only the base cases and the images of $t_k^n(p^\ell)$ which occur in the defining recurrence relations. So for $n = 2$ or 3, we need only the images of $t_k^n(p)$, while for $n = 4$, we need the images of $t_k^4(p)$ as well as $t_k^4(p^2)$ for $k = 1, 2$ which were the examples computed in a previous section.

By Lemma 2.21 of [2], we have that $T_k^n(p) = p^{k(k+1)/2}\pi_k^n(p)$, and from our earlier examples we obtained:

$$T_1^4(p^2) = p^2(\Gamma \operatorname{diag}(p^2,1,1,1)\Gamma + \Gamma \operatorname{diag}(p,p,1,1)\Gamma$$
$$T_2^4(p^2) = p^6\left[(p^4+p^2+1)\Gamma \operatorname{diag}(p,p,p,p)\Gamma + \Gamma \operatorname{diag}(p^2,p,p,1)\Gamma + \Gamma \operatorname{diag}(p^2,p^2,1,1)\Gamma\right]$$

which, together with the recursion relations of the last section, characterizes the structure of these new Hecke operators for GL_4.

Remark 6.1. *It may occur to the reader that since the $t_k^n(p^\ell)$ are symmetric polynomials, they are polynomials in the elementary symmetric polynomials ($t_k^n(p)$), and that the inverse image of the elementary symmetric polynomials is known, so why go through the trouble we have? The point is that by taking this later approach, even the generators (e.g. $t_1^4(p^2)$ and $t_2^4(p^2)$) of our classical Hecke algebra would only be defined as algebraic combinations of double cosets instead of linear combinations, and it was the cumbersome nature of computing products of double cosets which led us to the polynomial setting in the first place. Of course, defining the algebra in terms of recursion formulas forces one to consider products, but perhaps it is slightly more esthetic to have at least the generators characterized simply.*

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