Hecke Operators for GL_n and Buildings

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Abstract

We describe a representation of the local Hecke algebra for GL_n in which the Hecke operators act on the vertices of the Bruhat-Tits building for $SL_n(\mathbb{Q}_p)$. We also give a geometric interpretation of this representation, characterizing the action of our operators on a vertex in terms of the endpoints of minimal walks in the building. This generalizes work of Serre who defined Hecke operators acting on the vertices of a tree (the building for $SL_2(\mathbb{Q}_p)$).

1 Introduction

For various linear groups, there is a representation of the local Hecke algebra in which the representation space can be identified with a ring of polynomials invariant under the action of an associated Weyl group [5]. This representation has been exploited in the case of GL_n [3] to define new families of Hecke operators the sum of whose generating series has an especially nice expression as a rational function. In this paper, we extend Serre's work on trees [7] by defining a natural representation of the local Hecke algebra for GL_n in which the Hecke operators act on the vertices of the Bruhat-Tits building for $SL_n(\mathbb{Q}_p)$. We also give a specific geometric characterization (Theorem 2.3) showing that the endpoints of minimal walks on the (1-subcomplex of the) building correspond to the action of a sum of Hecke operators.

Fix a prime p, let $\Gamma = GL_n(\mathbb{Z}_p)$, and consider the local Hecke algebra $\mathcal{H}_p$ generated by all double cosets of the form $\Gamma \operatorname{diag}(p^{i_1}, \ldots, p^{i_n})\Gamma$, for integers $i_1, \ldots, i_n$. The integral Hecke algebra $\underline{\mathcal{H}}_p$ is generated by the n Hecke operators

$$T_k^n(p) = \Gamma \operatorname{diag}(\underbrace{1, \ldots, 1}_{n-k}, \underbrace{p, \ldots, p}_{k})\Gamma, \quad k = 1, \ldots, n$$

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and the full Hecke algebra $\mathcal{H}_p$ is generated by the n elements above, plus the element $T_n^n(p)^{-1}$.

In the next section we give a representation of $\mathcal{H}_p$ in which the Hecke operators act on the vertices of the building for $SL_n(\mathbb{Q}_p)$.

2 Hecke operators on Buildings

Before we define the representation, we briefly remind the reader about buildings. The Bruhat-Tits Building for $SL_n(\mathbb{Q}_p)$, Δ_n, is an $(n-1)$-dimensional simplicial complex whose vertices are homothety classes of lattices in a fixed n-dimensional vector space V over $\mathbb{Q}_p$ (see [2] or [4]). One defines an incidence relation on the vertices and the resulting flag complex is the building. Of particular interest are the apartments in the building whose structure is determined by the affine Coxeter group $\widetilde{A}_n$. We will exploit the structure of an apartment later to give a geometric characterization of our Hecke operators acting on the building.

To set up the representation, we need to talk about the correspondence between lattices in V and double cosets in $\mathcal{H}_p$. We begin a bit more generally.

2.1 Lattices and elementary divisors

Let E be a global or local field, $\mathcal{O}$ its ring of integers, and U an n-dimensional vector space over E. Let $\Gamma = GL_n(\mathcal{O})$ and $G = GL_n(E)$. We need recall some basic facts about lattices and elementary divisors. To that end, we assume that $\mathcal{O}$ is a PID (e.g., if E is any local field or a global field of class number one), and let $S = E^\times/\mathcal{O}^\times$. For $E = \mathbb{Q}$, we can take S to be the positive rationals, and for $E = \mathbb{Q}_p$, we let $S = \{p^\nu \mid \nu \in \mathbb{Z}\}$. The following facts follow from obvious modifications to §3.2 of [8].

Proposition 2.1. *With the notation as above,*

1. *Let $\xi \in GL_n(E)$. Then every double coset $\Gamma\xi\Gamma$ has a unique representative $\Gamma \operatorname{diag}(a_1, \ldots, a_n)\Gamma$ with $a_i \in S$, $a_i \mid a_{i+1}$ for all $1 \le i \le n$.*

2. *Let $\mathcal{L}$ be a lattice of rank n in U. Then Γ can be identified with the stabilizer in G of $\mathcal{L}$. It follows that for $\xi_1, \xi_2 \in G$ that $\Gamma\xi_1 = \Gamma\xi_2$ if and only if $\mathcal{L}\xi_1 = \mathcal{L}\xi_2$.*

3. *For two rank n lattices $\mathcal{L}$ and $\mathcal{M}$ in U, denote by $\{\mathcal{L} : \mathcal{M}\}$ the set of elementary divisors of $\mathcal{M}$ in $\mathcal{L}$. If $\{\mathcal{L} : \mathcal{M}\} = \{a_1, \ldots, a_n\}$ ($a_i \in S$), this means there exists an $\mathcal{O}$-basis $\{e_1, \ldots, e_n\}$ of $\mathcal{L}$ so that $\{a_1e_1, \ldots, a_ne_n\}$ is an $\mathcal{O}$-basis of $\mathcal{M}$. Then for lattices $\mathcal{L}, \mathcal{M}, \mathcal{N}$ of U, we have that $\{\mathcal{L} : \mathcal{M}\} = \{\mathcal{L} : \mathcal{N}\}$ if and only if there exists $\xi \in \Gamma$ with $\mathcal{M} = \mathcal{N}\xi$.*

4. *Let $\xi \in G$ and $\Gamma\xi\Gamma = \Gamma \operatorname{diag}(a_1, \ldots, a_n)\Gamma$. Then the correspondence $\Gamma\xi_\nu \mapsto \mathcal{L}\xi_\nu$ gives a one-to-one correspondence between the right cosets $\Gamma\xi_\nu \subset \Gamma\xi\Gamma$ and lattices $\mathcal{M}$ in U with $\{\mathcal{L} : \mathcal{M}\} = \{a_1, \ldots, a_n\}$.*

2.2 An action of the Hecke Algebra on the building for SL_n

Let V and Δ_n be as above. For a rank n lattice L in V (free $\mathbb{Z}_p$-module of rank n), denote by $[L]$ its homothety class, i.e., $[L] = \{M \mid M = \alpha L, \alpha \in \mathbb{Q}_p^\times\}$. The vertices of the building, $Vert(\Delta_n)$, are in one-to-one correspondence with homothety classes of lattices in V of rank n, and we let $\mathcal{B}$ denote the rational vector space with basis $Vert(\Delta_n)$.

For $\xi \in GL_n(\mathbb{Q}_p)$, and $\Gamma = GL_n(\mathbb{Z}_p)$, any double coset $\Gamma\xi\Gamma$ can be represented by a diagonal element, so we assume that henceforth. Let $\xi = \mathrm{diag}(p^{a_1}, \ldots, p^{a_n}) \in GL_n(\mathbb{Q}_p)$, and write the double coset $\Gamma\xi\Gamma$ as the disjoint union of a collection of right cosets $\{\Gamma\xi_\nu\}$. If L is a lattice in V of which Γ is the stabilizer, then by Proposition 2.1, the right cosets, $\{\Gamma\xi_\nu\}$, are in one-to-one correspondence with the collection of lattices $\{M\}$ for which $\{L : M\} = \{p^{a_1}, \ldots, p^{a_n}\}$. Given that the normal action of double cosets on an automorphic form is to sum over the action by the right cosets determined by the double coset, it is natural (using the notation above) to define the operator $T_{\mathcal{B}}(\xi) = T_{\mathcal{B}}(p^{a_1}, \ldots, p^{a_n}) \in \mathrm{End}(\mathcal{B})$ induced by:

$$T_{\mathcal{B}}(\xi)([L]) = T_{\mathcal{B}}(p^{a_1}, \ldots, p^{a_n})([L]) = \sum_{\xi_\nu}[L\xi_\nu] = \sum_{\{L:M\}=\{p^{a_1},\ldots,p^{a_n}\}}[M]$$

To ease the notation, when $\xi = \mathrm{diag}(p^{a_1}, \ldots, p^{a_n})$, we usually write $T_{\mathcal{B}}(\xi)([L]) = \sum_{\{L:M\}=\xi}[M]$ for $\sum_{\{L:M\}=\{p^{a_1},\ldots,p^{a_n}\}}[M]$.

2.3 The representation

In this section we show that the correspondence $\Gamma\xi\Gamma \mapsto T_{\mathcal{B}}(\xi)$ is (essentially) a faithful representation of the local Hecke algebra.

Let $\xi_1 = \mathrm{diag}(p^{a_1}, \ldots, p^{a_n})$, $\xi_2 = \mathrm{diag}(p^{b_1}, \ldots, p^{b_n})$ be elements of $GL_n(\mathbb{Q}_p)$ and write $\Gamma\xi_1\Gamma$ as the disjoint union $\cup\Gamma\alpha_i$, and write $\Gamma\xi_2\Gamma$ as the disjoint union $\cup\Gamma\beta_j$. In the Hecke algebra $\mathcal{H}_p$, the multiplication law is defined by (e.g., see section 3.1 of [1]):

$$(\Gamma\xi_1\Gamma)(\Gamma\xi_2\Gamma) = \Gamma\xi_1\Gamma\xi_2\Gamma = \sum_{i,j}\Gamma\alpha_i\beta_j$$

where the right cosets are not necessarily distinct. More precisely,

$$(\Gamma\xi_1\Gamma)(\Gamma\xi_2\Gamma) = \sum_{i,j}\Gamma\alpha_i\beta_j = \sum_{\Gamma\xi\Gamma}c(\xi)\Gamma\xi\Gamma$$

where the sum is over all double cosets $\Gamma\xi\Gamma \subset \Gamma\xi_1\Gamma\xi_2\Gamma$, and where $c(\xi)$ is the number of pairs (i, j) for which $\Gamma\alpha_i\beta_j = \Gamma\xi$.

Theorem 2.1. *With the notation as above, the correspondence* $\Gamma\xi\Gamma \mapsto T_{\mathcal{B}}(\xi)$ *induces a representation* $\Psi : \mathcal{H}_p \to \mathrm{End}(\mathcal{B})$*, whose kernel consists of double cosets of the form* $\Gamma\xi\Gamma$ *with* $\xi = p^\mu I_n$*,* $\mu \in \mathbb{Z}$*.*

Proof. The correspondence induces a linear map Ψ, so we first verify that Ψ is a ring homomorphism. Using the notation above, we have

$$\begin{aligned} T_{\mathcal{B}}(\xi_1)T_{\mathcal{B}}(\xi_2)([L]) &= T_{\mathcal{B}}(\xi_1)(\sum_{\{L:M\}=\xi_2}[M]) \\ &= \sum_{\{L:M\}=\xi_2}\sum_{\{M:N\}=\xi_1}[N] \end{aligned}$$

Recall that $\Gamma\xi_1\Gamma = \cup\Gamma\alpha_i$, $\Gamma\xi_2\Gamma = \cup\Gamma\beta_j$. By Proposition 2.1, each lattice M for which $\{L : M\} = \xi_2$ is of the form $M = L\beta_j$. Now

$$\{M : N\} = \xi_1 \iff \{L\beta_j : N\} = \xi_1 \iff \{L : N\beta_j^{-1}\} = \xi_1$$

Now let P be such that $\{L : P\} = \xi_1$. Then again by Proposition 2.1, $P = L\alpha_i$ for some i. But then $P = N\beta_j^{-1}$, so $N = P\beta_j = L\alpha_i\beta_j$.

Thus, $T_{\mathcal{B}}(\xi_1)T_{\mathcal{B}}(\xi_2)([L]) = \sum_{\{L:M\}=\xi_2}\sum_{\{M:N\}=\xi_1}[N] = \sum_{i,j}[L\alpha_i\beta_j]$. From the discussion preceding the theorem (and once again Proposition 2.1), this last sum is exactly $\sum_{\Gamma\xi\Gamma} c(\xi)T_{\mathcal{B}}(\xi)([L])$ which is the image of $(\Gamma\xi_1\Gamma)(\Gamma\xi_2\Gamma)$.

To compute the kernel of Ψ, suppose $\sum_{\Gamma\xi\Gamma} c(\xi)T_{\mathcal{B}}(\xi)$ is the trivial map. Then

$$\sum_{\Gamma\xi\Gamma} c(\xi)T_{\mathcal{B}}(\xi)([L]) = \sum_{\Gamma\xi\Gamma} c(\xi)\sum_{\{L:M\}=\xi}[M] = [L]$$

for all vertices $[L] \in \mathrm{Vert}(\Delta_n)$. But the elements $[M] \in \mathrm{Vert}(\Delta_n)$ are a basis for $\mathcal{B}$, so we have only one ξ, and for that ξ, $c(\xi) = 1$. Thus we have $\sum_{\{L:M\}=\xi}[M] = [L]$ for all $[L]$. Now if $\Gamma\xi\Gamma = \cup\Gamma\xi_\nu$, then by Proposition 2.1, $\sum_{\{L:M\}=\xi}[M] = \sum_\nu[L\xi_\nu] = [L]$, so there can be only one right coset: $\Gamma\xi\Gamma = \Gamma\xi$, and $[L\xi] = [L]$. Since $\{L : L\xi\} = \xi$, we must have $\xi = p^\mu I_n$ for some integer μ. $\square$

2.4 A comparison to Serre's operators on trees

In this section, we observe that the operators defined above are natural generalizations of those defined by Serre [7] on trees. In particular, we show that when $n = 2$, they can be combined to produce an operator which coincides with Serre's. In the following section, we give a detailed interpretation of the higher rank operators, and connect them to walks on the building.

Since the action of $T_n^n(p)$ is trivial, we may restrict our attention to the integral local Hecke algebra. To that end, we need only consider double cosets $\Gamma\,\mathrm{diag}(p^{i_1}, \ldots, p^{i_n})\Gamma$ with the i_k nonnegative integers. For positive integers n and ℓ, denote by $P_n(\ell)$ the set of partitions of ℓ into n non-negative pieces. A typical element of $P_n(\ell)$ is denoted $\mathbf{i} = (i_1, \dots, i_n)$ with $0 \le i_1 \le i_2 \le \cdots \le i_n$ and $\sum i_k = m$.

Using the operators already defined, we define more familiar, but less discriminating, operators which sum all of the basic Hecke operators corresponding to the partitions in $P_n(\ell)$:

$$T_{\mathcal{B}}(p^\ell) = \sum_{\mathbf{i} \in P_n(\ell)} T_{\mathcal{B}}(p^{i_1}, \dots, p^{i_n}).$$

We recall that V is our fixed n-dimensional vector space over $\mathbb{Q}_p$. By the elementary divisor theorem, given two rank n lattices $\mathcal{L}$ and $\mathcal{M}$ in V, there exists a basis $\{e_1, \dots, e_n\}$ of V and rational integers $a_1, \dots, a_n$, so that

$$\mathcal{L} = \mathbb{Z}_p e_1 \oplus \cdots \oplus \mathbb{Z}_p e_n, \qquad \mathcal{M} = \mathbb{Z}_p p^{a_1} e_1 \oplus \cdots \oplus \mathbb{Z}_p p^{a_n} e_n.$$

Given this notation, Serre [7], in the case of $n = 2$, defines the distance between the two vertices represented by the lattices $\mathcal{L}$ and $\mathcal{M}$ as $|a_1 - a_2|$. Actually, this notion of distance is well-defined on homothety classes of lattices. Generalizations of the notion of distance for $n > 2$ appear in [6]. Using this distance function, Serre defines a graph letting the vertices be homothety classes of lattices and by placing an edge between any two vertices which are unit distance apart. It then follows that the resulting graph (which is the Bruhat-Tits building for $SL_2(\mathbb{Q}_p)$) is a $(p+1)$-regular tree, and that vertices which are distance m from a given vertex are simply the endpoints of walks without backtracking of length m.

Serre defines a family of $\mathbb{Z}$-linear operators θ_ℓ which act on the free abelian group generated by the vertices of the tree by setting

$$\theta_\ell(v) = \sum_{d(v,w)=\ell} w,$$

that is, a vertex is mapped to the sum of its neighbors at distance ℓ. From the structure of the tree, he notes that

$$\theta_1\theta_1 = \theta_2 + (p+1)\theta_0 \quad \text{and} \quad \theta_1\theta_\ell = \theta_{\ell+1} + p\theta_{\ell-1} \text{ for } \ell \ge 2.$$

Compared to the recursions satisfied by classical Hecke operators $T(p^\ell)$ (or merely in terms of simplicity of expression) the recursion is slightly off in the base case. To correct this, Serre defines new operators

$$T_0 = \theta_0, \quad T_1 = \theta_1, \quad \text{and} \quad T_\ell = \theta_\ell + T_{\ell-2} \text{ for } \ell \ge 2,$$

which yields (for all $\ell \ge 1$) the relation

$$T_1 T_\ell = T_{\ell+1} + pT_{\ell-1}.$$

While the sum of the generating series for the θ_ℓ operators is not quite as simple, the generating series for the T_ℓ operators sums to a rational function with a familiar form:

$$\sum_{\ell \ge 0} T_\ell u^\ell = \left[1 - T_1 u + pu^2\right]^{-1}.$$

Identifying the vertices of the graph with homothety classes of lattices, we rewrite Serre's operators as

$$\theta_\ell([\mathcal{L}]) = \sum_{d([\mathcal{L}],[\mathcal{M}])=\ell} [\mathcal{M}], \ \ell \geq 1; \quad \theta_0 = 1$$

$$T_\ell([\mathcal{L}]) = \theta_\ell([\mathcal{L}]) + T_{\ell-2}([\mathcal{L}]), \ \ell \geq 2; \quad T_0 = \theta_0, \ T_1 = \theta_1$$

and show that his operators T_ℓ agree with ours $T_\mathcal{B}(p^\ell)$:

Proposition 2.2. *For $n = 2$, we have $T_\ell = T_\mathcal{B}(p^\ell)$.*

Proof. By Serre's definition of distance,

$$d([\mathcal{L}], [\mathcal{M}]) = \ell \text{ iff } \{\mathcal{L} : \mathcal{M}\} = \{p^a, p^b\} \text{ with } \ell = |a - b|$$

Since $[\mathcal{M}] = [p^k\mathcal{M}]$ we have that

$$\theta_\ell([\mathcal{L}]) = \sum_{\{\mathcal{L}:\mathcal{M}\}=\{1,p^\ell\}} [\mathcal{M}]$$

Also, by definition, we have $T_\mathcal{B}(p^\ell) = \sum_{k=0}^{[\ell/2]} T_\mathcal{B}(p^k, p^{\ell-k})$.

To see $T_\ell = T_\mathcal{B}(p^\ell)$ we proceed by induction on ℓ. For $\ell = 0$ this is trivial. For $\ell = 1$, we observe that $T_1([\mathcal{L}]) = \theta_1([\mathcal{L}]) = \sum_{\{\mathcal{L}:\mathcal{M}\}=\{1,p\}}[\mathcal{M}] = T_\mathcal{B}(1, p)([\mathcal{L}]) = T_\mathcal{B}(p)([\mathcal{L}])$.

Now assume that $\ell \geq 2$.

$$\begin{aligned}
T_\ell([\mathcal{L}]) &= \theta_\ell([\mathcal{L}]) + T_{\ell-2}([\mathcal{L}]) \\
&= \sum_{\{\mathcal{L}:\mathcal{M}\}=\{1,p^\ell\}} [\mathcal{M}] + T_\mathcal{B}(p^{\ell-2})([\mathcal{L}]) \quad \text{by induction} \\
&= T_\mathcal{B}(1, p^\ell)([\mathcal{L}]) + \sum_{k=0}^{[(\ell-2)/2]} T_\mathcal{B}(p^k, p^{\ell-2-k})([\mathcal{L}]) \\
&= T_\mathcal{B}(1, p^\ell)([\mathcal{L}]) + \sum_{k=0}^{[(\ell-2)/2]} T_\mathcal{B}(p, p)T_\mathcal{B}(p^k, p^{\ell-2-k})([\mathcal{L}]) \\
&= T_\mathcal{B}(1, p^\ell)([\mathcal{L}]) + \sum_{k=0}^{[(\ell-2)/2]} T_\mathcal{B}(p^{k+1}, p^{\ell-1-k})([\mathcal{L}]) \\
&= T_\mathcal{B}(1, p^\ell)([\mathcal{L}]) + \sum_{k=1}^{[\ell/2]} T_\mathcal{B}(p^k, p^{\ell-k})([\mathcal{L}]) \\
&= \sum_{k=0}^{[\ell/2]} T_\mathcal{B}(p^k, p^{\ell-k})([\mathcal{L}]) = T_\mathcal{B}(p^\ell)([\mathcal{L}]). \quad \square
\end{aligned}$$

$\square$

2.5 Hecke Operators and Walks

The Bruhat-Tits building Δ_n is the union of subcomplexes known as apartments whose structure is completely specified by the affine Weyl group $\widetilde{A}_n$.

To specify an apartment in Δ_n, choose an unordered set of n one-dimensional subspaces $V_1, V_2, \ldots, V_n$, such that $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$. The vertices in the apartment can be viewed as homothety classes of lattices $L = \mathbb{Z}_p v_1 \oplus \cdots \oplus \mathbb{Z}_p v_n$ with $v_i \in V_i$ for each i. Now fix an ordering of the chosen subspaces V_i, which we will refer to as an *orientation* of the apartment. In the end, we will let the symmetric group, S_n, act on the apartment, so the situation will be independent of orientation. To begin, fix a vertex v in the apartment, and let v correspond to the class of the lattice $L_0 = \mathbb{Z}_p e_1 \oplus \cdots \oplus \mathbb{Z}_p e_n$. Then the vertices of the apartment are in one-to-one correspondence with the classes of lattices $\mathbb{Z}_p p^{a_1} e_1 \oplus \cdots \oplus \mathbb{Z}_p p^{a_n} e_n$, where the a_i run over $\mathbb{Z}$. Focusing attention on the ordered n-tuple $(a_1, a_2, \ldots, a_n)$, the vertices of the apartment are thus in one-to-one correspondence with the elements of $\mathbb{Z}^n / \mathbb{Z} \cdot (1, 1, \ldots, 1)$.

Following [2], if $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are in $\mathbb{Z}^n$, write

$$u \preceq v \text{ if } u_i \leq v_i \leq u_i + 1 \text{ for all } i.$$

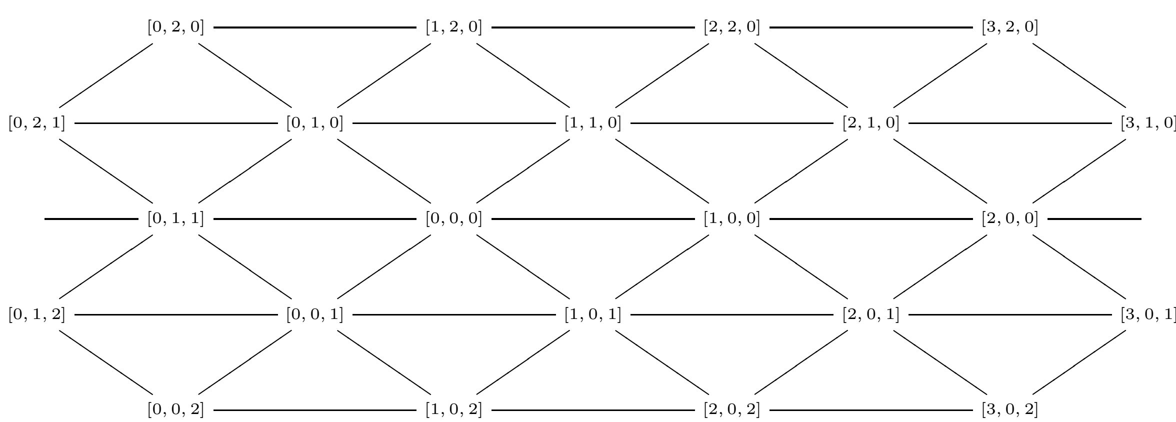
Calling two elements of $\mathbb{Z}^n/\mathbb{Z}\cdot(1, 1, \ldots, 1)$ incident if they admit representatives u and v with $u \preceq v$ produces a flag complex which defines the full simplicial structure of the apartment. Consider our fixed vertex v represented by the lattice $L_0 = \mathbb{Z}_p e_1 \oplus \cdots \oplus \mathbb{Z}_p e_n$. Any chamber (i.e., $(n-1)$-simplex) containing v corresponds to a maximal flag of lattices:

$$pL_0 = L_n \subsetneq L_{n-1} \subsetneq \cdots \subsetneq L_1 \subsetneq L_0.$$

Fix such a (fundamental) chamber, by choosing

$$L_i = \mathbb{Z}_p p e_1 \oplus \cdots \oplus \mathbb{Z}_p p e_i \oplus \mathbb{Z}_p e_{i+1} \oplus \cdots \oplus \mathbb{Z}_p e_n.$$

The rest of the apartment is labeled by letting the generators of the Coxeter group act on the fundamental chamber. For example, if we denote by $[a_1, \ldots, a_n]$ the class of the lattice $\mathbb{Z}_p p^{a_1} e_1 \oplus \cdots \oplus \mathbb{Z}_p p^{a_n} e_n$, we obtain the following labeling of a piece of an apartment for SL_3:



There is a natural group operation defined on the vertices as we have identified the vertices with the elements of $\mathbb{Z}^n/\mathbb{Z}\cdot(1,1,\ldots,1)$. Moreover there is a natural geometric interpretation as well. The vertices adjacent to $[0,0,\ldots,0]$ (for $n=3$ labeled counterclockwise $[1,0,0],[0,1,1],[0,1,0],[0,1,1],[0,0,1],[1,0,1]$) define directions in which to move within the apartment which is consistent with the group law: For example, moving from $[0,0,0]$ to $[1,0,0]$ and then in the direction indicated by $[0,1,0]$ brings us to the same vertex as moving from $[0,0,0]$ in the direction indicated by $[1,1,0]$. Thus we can think of a vertex $[a_1,\ldots,a_n]$ as the endpoint of a walk (along the 1-subcomplex of the apartment) from $[0,0,\ldots,0]$ to $[a_1,\ldots,a_n]$ which takes a_1 steps in the $[1,0,0,\ldots,0]$ direction, a_2 steps in the $[0,1,0,\ldots,0]$ direction, $\ldots$, and a_n steps in the $[0,\ldots,0,1]$ direction.

Now the geometric action of our Hecke operator $T_{\mathcal{B}}(p^{i_1},\ldots,p^{i_n})$ becomes a bit clearer. Recall

$$T_{\mathcal{B}}(p^{a_1},\ldots,p^{a_n})([L]) = \sum_{\{L:M\}=\{p^{a_1},\ldots,p^{a_n}\}} [M],$$

so at least restricted to an apartment, we see that the sum consists of those vertices $[a_{\sigma(1)},a_{\sigma(2)},\ldots,a_{\sigma(n)}]$ for σ in the symmetric group S_n, that is the endpoints of walks proceeding $a_{\sigma(i)}$ units in the ith "standard" direction.

The interpretation of $T_{\mathcal{B}}(p^{a_1},\ldots,p^{a_n})$ on the building, Δ_n, is a bit more complicated. By a minimal walk between two vertices, we simply mean a walk (a sequence of vertices $\{v_1,\ldots,v_m\}$ each pair $\{v_i,v_{i+1}\}$ connected by an edge), between the two vertices which is of minimal length. We characterize the endpoints of minimal walks in the building in the following theorem.

Theorem 2.3. *Let $v_0=[L]$ be a vertex in the Bruhat-Tits building Δ_n for $SL_n(\mathbb{Q}_p)$ which is represented by the homothety class of the lattice L. The set of vertices in the building which are endpoints of minimal walks of length m from v_0 is*

$$\sum_{0\le k_1\le\cdots\le k_{n-2}\le m} T_{\mathcal{B}}(1,p^{k_1},\ldots,p^{k_{n-2}},p^m)([L]).$$

Proof. Consider a minimal walk, γ, between two vertices v_0 and v_m in Δ_n. Denote the walk by the sequence of vertices through which it passes: $\gamma=\{v_0,v_1,\cdots,v_m\}$. Choose chambers C_0 and C_m with $v_0\in C_0$ and $v_m\in C_m$, and let A be an apartment containing the chambers C_0 and C_m. Finally, let $\rho=\rho_{A,C_0}$ be the canonical retraction of Δ_n onto A centered at C_0.

Since the retraction ρ is a simplicial map, it takes the walk γ to another walk $\rho(\gamma)=\{\rho(v_0),\rho(v_1),\ldots,\rho(v_{m-1}),\rho(v_m)\}$ contained in A. But v_0 and v_m are both in A, so are fixed pointwise by ρ, making $\rho(\gamma)$ a walk in A from v_0 to v_m. Moreover, it is clear that $\rho(\gamma)$ has length at most m, since it has at most $m+1$ distinct vertices defining the walk. Finally, since m is the length of any minimal walk from v_0 to v_m, we must have that $\rho(\gamma)$ has length m, and hence is a minimal walk in A from v_0 to v_m.

Since our interest is only to count the endpoints of minimal walks of length m, we may assume from the argument above that any such walk is wholly contained in an apartment.

Thus we need only characterize the vertices of an apartment which are the endpoints of minimal walks (in that apartment) of length m. Recall that there is a one-to-one correspondence between the vertices of an apartment and elements in $\mathbb{Z}^n/\mathbb{Z}(1,1,\ldots,1)$. Let v be a vertex in a fixed apartment A, and without loss normalize v as $[a_1,\ldots,a_n]$ where some $a_i = 0$ and all $a_j \geq 0$.

We claim that v is the endpoint of a minimal walk of length $\max\{a_1,\ldots,a_n\}$, starting from $[0,\ldots,0]$. By the action of the Weyl group on the apartment (in this case the Weyl group is the symmetric group S_n), there is no loss in assuming that $v = [a_1,\ldots,a_n]$ where $0 = a_1 \leq a_2 \leq \cdots \leq a_n$. Fixing a fundamental chamber with vertices $(0,\ldots,0)$, $[0,\ldots,0,1]$, $\ldots$, $[0,1,\ldots,1]$ defines the "directions" in which one may walk, so the walk is achieved simply by proceeding a_2 units in the $[0,1,\ldots,1]$ direction, followed by $a_3 - a_2$ units in the $[0,0,1,\ldots,1]$ direction, and so on until $a_n - a_{n-1}$ units in the $[0,\ldots,0,1]$ direction. It is clear no walk can reach the vertex in fewer than a_n steps, so this walk is minimal.

For a vertex v, denote by v^{S_n} the orbit of v under the action of the symmetric group. Then in a given apartment, the endpoints of minimal walks of length m starting from $[0,0,\ldots,0]$ is given by

$$\sum_{0\leq k_1\leq\cdots\leq k_{n-2}\leq m} [0,k_1,\ldots,k_{n-2},m]^{S_n}.$$

The identification of the vertices of an apartment with the elements of $\mathbb{Z}^n/\mathbb{Z}(1,\ldots,1)$ has also been characterized earlier in terms of elementary divisors, namely if $[0,\ldots,0] = [L]$, then $[a_1,\ldots,a_n]$ represents the class of a lattice M with elementary divisors $\{L : M\} = \{p^{a_1},\ldots,p^{a_n}\}$, and conversely. From this, the theorem follows immediately. $\square$

Remark 2.4. *Note that in the case $n = 2$, $\sum_{0\leq k_1\leq\cdots\leq k_{n-2}\leq m} T_{\mathcal{B}}(1,p^{k_1},\ldots,p^{k_{n-2}},p^m)$ reduces to $T_{\mathcal{B}}(1,p^m) = \theta_m$, Serre's original operator on the tree.*

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