Split Orders and Convex Polytopes in Buildings

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Abstract

As part of his work to develop an explicit trace formula for Hecke operators on congruence subgroups of $SL_2(\mathbb{Z})$, Hijikata [4] defines and characterizes the notion of a split order in $M_2(k)$, where k is a local field. In this paper, we generalize the notion of a split order to $M_n(k)$ for $n > 2$ and give a natural geometric characterization in terms of the affine building for $SL_n(k)$. In particular, we show that there is a one-to-one correspondence between split orders in $M_n(k)$ and a collection of convex polytopes in apartments of the building such that the split order is the intersection of all the maximal orders representing the vertices in the polytope. This generalizes the geometric interpretation in the $n = 2$ case in which split orders correspond to geodesics in the tree for $SL_2(k)$ with the split order given as the intersection of the endpoints of the geodesic.

1 Introduction

The study of orders in noncommutative algebras has a long history with known applications to class field theory, modular forms, and geometry. In [4], Hijikata defines and characterizes split orders in $M_2(k)$, k a local field, as part of his work to develop an explicit trace formula for Hecke operators on congruence subgroups of $SL_2(\mathbb{Z})$. His characterization of split orders is entirely algebraic, characterizing them as either maximal orders or the intersection of two uniquely determined maximal orders. More precisely, he shows that

Proposition 1.1. *Let k be a local field, $\mathcal{O}$ its valuation ring, and $\mathfrak{p}$ the unique maximal ideal of $\mathcal{O}$. Let S be an $\mathcal{O}$-order in $A = M_2(k)$; the following are equivalent and define the notion of a* split order *in A.*

1. *S contains a subset which is $A^\times$-conjugate to $\left(\begin{smallmatrix} \mathcal{O} & 0 \\ 0 & \mathcal{O} \end{smallmatrix}\right)$.*
2. *S is $A^\times$-conjugate to $\left(\begin{smallmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p}^\nu & \mathcal{O} \end{smallmatrix}\right)$ for some non-negative integer ν.*
3. *S is the intersection of at most two maximal orders in A.*

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4. S is either maximal or the intersection of two uniquely determined distinct maximal orders.

Hijikata's proposition has the following geometric interpretation. For k a local field, the vertices of the affine building associated to $SL_2(k)$ are in one-to-one correspondence with the maximal orders in $M_2(k)$. Moreover, it is well-known that SL_2-building is actually a $(q+1)$-regular tree (q the cardinality of the residue field of k), so that any two vertices determines a unique path or geodesic between them. Thus split orders are in one-to-one correspondence with the geodesics of finite (nonnegative) length on the tree, with the split order realized as the intersection of the maximal orders representing the endpoints of the geodesic; geodesics of length zero are the maximal orders.

In this paper we consider the generalization of the notion of a split order to $B = M_n(k)$ for $n > 2$, and give a geometric characterization using the affine building for $SL_n(k)$. We take as a definition of a split order, an order in $M_n(k)$ which contains a subring $B^\times$-conjugate to $R = \begin{pmatrix} \mathcal{O} & & 0 \\ & \ddots & \\ 0 & & \mathcal{O} \end{pmatrix}$, hereafter denoted as $R = \mathrm{diag}(\mathcal{O}, \ldots, \mathcal{O})$. The geometric generalization which we derive agrees with the $n = 2$ characterization, though in a manner slightly more nuanced than the one given above. In generalizing, one finds that there is no comparable uniqueness statement (as in Hijikata's proposition) which characterizes a split order as the intersection of a uniquely determined minimal set of maximal orders. Rather, the uniqueness arises by considering the set of all maximal orders which contain the split order. We show that

- there is an apartment which contains the set of all maximal orders containing a given split order,
- this collection of maximal orders consists of the set of all vertices which lie in a convex polytope uniquely determined by the split order, and
- the split order is the intersection of all the maximal orders in this convex polytope.

In the case $n = 2$ (where the building is a tree), Hijikata's result shows that a split order is the intersection of the maximal orders which are the endpoints of the geodesic which characterize it; from this work it follows that the split order is also the intersection of all the maximal orders contained in the geodesic. For $n > 2$ and in the case where all the maximal orders are vertices of a single chamber in the building, the notion of split orders reduces to that of chain orders studied (to a different end) in [1]. In that case the notion of convexity is implicit in the structure of the building as the convex polytopes which arise are simply faces of the chamber. The present work addresses finitely many maximal orders chosen arbitrarily in any apartment of the building. The fact that there is a given apartment containing all the maximal orders which contain a split order is an interesting extension of the standard building fact that any two simplicies in a building are contained in a single apartment, and may point to more complicated structure implicit in the building.

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2 Split Orders

2.1 Definition and initial characterization

Let k be a local field, $\mathcal{O}$ its valuation ring, and $\mathfrak{p} = \pi\mathcal{O}$ the unique maximal ideal of $\mathcal{O}$, with π a fixed uniformizing parameter. Let B be the central simple algebra $M_n(k)$, and fix a subring R having the form $R = \mathrm{diag}(\mathcal{O}, \ldots, \mathcal{O})$. Recall that an order $S \subset B$ is a subring of B containing the identity which is also a free $\mathcal{O}$-module having rank n^2. We begin our investigation of split orders with the special case in which the order $S \subset B$ actually contains the subring R. We shall see that the consideration of general split orders (containing a conjugate of R) simply amounts to a change of basis and shifts the geometric perspective from one apartment to another.

We first give an initial, though somewhat unsatisfying, algebraic characterization of these split orders. Let $E^{(i,j)}$ denote the $n \times n$ matrix with a 1 in the (i, j) position and zeros elsewhere.

Proposition 2.1. *1. Let $S \subset M_n(k)$ be a ring containing $E^{(i,i)}$ for $1 \leq i \leq n$. Then $A = (a_{ij}) \in S$ if and only if $a_{ij}E^{(i,j)} \in S$ for all i, j.*

2. Let S be an order in $M_n(k)$ containing $E^{(i,i)}$ for $1 \leq i \leq n$. Then S has the form $S = \begin{pmatrix} \mathcal{O} & & \mathfrak{p}^{\nu_{ij}} \\ & \ddots & \\ \mathfrak{p}^{\nu_{ij}} & & \mathcal{O} \end{pmatrix}$ which we simplify to $S = (\mathfrak{p}^{\nu_{ij}})$ with the understanding that $\nu_{ii} = 0$ for all i.

3. Let $S = (\mathfrak{p}^{\nu_{ij}}) \subset M_n(k)$ be a set with $\nu_{ii} = 0$ for all i. Then S is an order if and only if $\nu_{ik} + \nu_{kj} \geq \nu_{ij}$ for every i, j, k.

Proof. For the first item, one direction is obvious and for the other, simply observe that $E^{(i,i)}AE^{(j,j)} = a_{ij}E^{(i,j)}$. For (2), let $S_{ij} = \{E^{(i,i)}AE^{(j,j)} = a_{ij}E^{(i,j)} \mid A \in S\}$. Since S is an order and hence has rank n^2 as an $\mathcal{O}$-module, it follows that $S_{ij} \neq \{0\}$. Since S contains all the $E^{(i,i)}$, it is obvious that S_{ij} is a fractional $\mathcal{O}$-ideal, hence has the form $\mathfrak{p}^{\nu_{ij}}E^{(i,j)}$. Since $\mathcal{O}E^{(i,i)} \subseteq S_{ii}$, it is easy to deduce (e.g., from the integrality of elements of S [6]) that $S_{ii} = \mathcal{O}E^{(i,i)}$. For (3), if S is closed under multiplication, then $S_{ik}S_{kj} \subseteq S_{ij}$, hence $\mathfrak{p}^{\nu_{ik}}\mathfrak{p}^{\nu_{kj}} \subseteq \mathfrak{p}^{\nu_{ij}}$, so $\nu_{ik} + \nu_{kj} \geq \nu_{ij}$. Conversely a set $S = (\mathfrak{p}^{\nu_{ij}})$ with $\nu_{ii} = 0$ is an order if and only if it is closed under multiplication. Let $A = \sum_{i,j} a_{ij}E^{(i,j)}$, $B = \sum_{k,\ell} b_{k\ell}E^{(k,\ell)} \in S$. Now $AB = \sum_{i,j,k,\ell} a_{ij}b_{k\ell}E^{(i,j)}E^{(k,\ell)} = \sum_{i,j,\ell} a_{ij}b_{j\ell}E^{(i,j)}E^{(j,\ell)} = \sum_{i,\ell}\left(\sum_j a_{ij}b_{j\ell}\right)E^{(i,\ell)}$. Since $a_{ij} \in \mathfrak{p}^{\nu_{ij}}$, and $b_{j\ell} \in \mathfrak{p}^{\nu_{j\ell}}$, the condition $\nu_{ij} + \nu_{j\ell} \geq \nu_{i\ell}$ shows that $\sum_j a_{ij}b_{j\ell} \in \mathfrak{p}^{\nu_{i\ell}}$, and hence $AB \in S$. ∎

2.2 The maximal orders which contain a split order

Next we consider the extent to which the alternate characterizations of split orders in $M_2(k)$ given by Hijikata hold in $B = M_n(k)$ when $n > 2$. Naive conjectures concerning a minimal set of maximal orders whose intersection produces the split order are easily shown not to hold in general, however a uniqueness statement can be deduced characterizing split orders as the intersection of a geometrically distinguished collection of maximal orders which nicely generalizes the situation for $n = 2$.

In particular, we consider whether a split order is characterized by the set of all maximal orders which contain it. To that end, we let $\Lambda_0 = M_n(\mathcal{O})$ be a fixed maximal order in B. It is well known [6] that every maximal order in B is conjugate by an element of $B^\times$ to Λ_0.

We first characterize those maximal orders which contain the subring R, which reduces to characterizing those $\xi = B^\times$, so that $R \subset \xi^{-1}\Lambda_0\xi$. Since $M_n(k) = k^\times M_n(\mathcal{O})$ and the action by conjugation of $k^\times$ is trivial, we may assume that $\xi \in M_n(\mathcal{O})$, and in particular, we may choose for ξ any representative of $GL_n(\mathcal{O})\xi$. Thus there is no loss of generality to assume that ξ is in Hermite normal form (see e.g., [5]), that is $\xi = \begin{pmatrix} \pi^{m_1} & a_{12} & & \dots & & a_{1n} \\ 0 & \pi^{m_2} & a_{23} & \dots & & a_{2n} \\ 0 & 0 & \ddots & & & \\ 0 & 0 & & \dots & \pi^{m_{n-1}} & a_{n-1\,n} \\ 0 & 0 & & \dots & & \pi^{m_n} \end{pmatrix}$, an upper triangular matrix with powers of the fixed uniformizer on the diagonal and entries a_{ij} $(i < j)$ in a fixed set of residues of $\mathcal{O}/\pi^{m_j}\mathcal{O}$. We may and do assume the representative of the zero class is actually zero.

Proposition 2.2. *With the notation and assumptions as above, we have the $R \subset \xi^{-1}\Lambda_0\xi$ if and only if ξ is diagonal, $\xi = \operatorname{diag}(\pi^{m_1}, \dots, \pi^{m_n})$.*

Proof. We show that $\xi R \xi^{-1} \subset \Lambda_0$ if and only if $\xi = \operatorname{diag}(\pi^{m_1}, \dots, \pi^{m_n})$. If ξ is diagonal, the result is clear, so we assume that ξ is in Hermite normal form and deduce inductively that the off-diagonal entries are zero.

Let $D = \operatorname{diag}(d_1, \dots, d_n) \in R$, and consider $C = \xi D \xi^{-1}$. We need to examine explicitly the entries of C. Obviously $\xi D = \begin{pmatrix} \pi^{m_1}d_1 & a_{12}d_2 & & \dots & & a_{1n}d_n \\ 0 & \pi^{m_2}d_2 & a_{23}d_3 & \dots & & a_{2n}d_n \\ 0 & 0 & \ddots & & & \\ 0 & 0 & & \dots & \pi^{m_{n-1}}d_{n-1} & a_{n-1\,n}d_n \\ 0 & 0 & & \dots & & \pi^{m_n}d_n \end{pmatrix}$, and $C_{ij} = \sum_{k=1}^n (\xi D)_{ik}(\xi^{-1})_{kj} = \sum_{k=i}^j (\xi D)_{ik}(\xi^{-1})_{kj}$, since both ξD and ξ^{-1} are upper triangular. Here as is standard $(\xi^{-1})_{kj} = (\det \xi)^{-1}(-1)^{k+j} \det \xi(j \mid k)$ where $\xi(j \mid k)$ is the $(n-1)\times(n-1)$ minor obtained by deleting the jth row and kth column of ξ.

For $1 \le i < n$ we consider the entry $C_{i\,i+1} = \sum_{k=i}^{i+1} (\xi D)_{ik}(\xi^{-1})_{k\,i+1}$. We compute

$$(\xi^{-1})_{i\,i+1} = (\det \xi)^{-1}(-1)^{2i+1} \det \begin{pmatrix} \pi^{m_1} & & & \dots & * & \\ & \ddots & & & & * \\ & & \pi^{m_{i-1}} & a_{i-1\,i+1} & \dots & \\ & & & a_{i\,i+1} & a_{i\,i+2} & \dots \\ & 0 & & & \pi^{m_{i+2}} & \\ & & & 0 & & \ddots \end{pmatrix} = \frac{-a_{i\,i+1}}{\pi^{m_i+m_{i+1}}},$$

so $C_{i\,i+1} = (\pi^{m_i} d_i)(\frac{-a_{i\,i+1}}{\pi^{m_i+m_{i+1}}}) + (a_{i\,i+1}d_{i+1})(\pi^{-m_{i+1}\,i+1}) = \frac{a_{i\,i+1}}{\pi^{m_{i+1}}}(d_{i+1}-d_i)$. Since C must be an element of $\Lambda_0 = M_n(\mathcal{O})$ we must have $C_{i\,i+1} \in \mathcal{O}$. Since the d_i's are arbitrary we may assume that $\pi \nmid (d_{i+1}-d_i)$, so $C_{i\,i+1} = \frac{a_{i\,i+1}}{\pi^{m_{i+1}}}(d_{i+1}-d_i) \in \mathcal{O}$ forces $a_{i\,i+1} \equiv 0 \pmod{\pi^{m_{i+1}}}$. But we have chosen ξ in Hermite normal form which forces $a_{i\,i+1} = 0$.

Inductively, suppose $a_{ij} = 0$ for $i+1 \le j \le i+\ell$. We show $a_{i\,i+\ell+1} = 0$. Consider the entry

$$C_{i,i+\ell+1} = \sum_{k=i}^{i+\ell+1} (\xi D)_{ik}(\xi^{-1})_{k\,i+\ell+1} = (\xi D)_{ii}(\xi^{-1})_{i,i+\ell+1}(\xi D)_{i,i+\ell+1}(\xi^{-1})_{i+\ell+1,i+\ell+1},$$

since $(\xi D)_{i,i+r} = a_{ir}d_r = 0$ for $1 \le r \le \ell$. As before, there is only one term at issue, $(\xi^{-1})_{i,i+\ell+1} = (\det\xi)^{-1}(-1)^{2i+\ell+1}\det\xi(i+\ell+1\,|\,i)$. Now the minor has the form:

$$\xi(i+\ell+1\,|\,i) = \begin{pmatrix} \ddots & & & & & & & \\ & \pi^{m_{i-1}} & a_{i-1,i+1} & \cdots & & & & \\ & & a_{i,i+1} & a_{i,i+2} & a_{i,i+3} & \cdots & a_{i,i+\ell+1} & \cdots \\ & & \pi^{m_{i+1}} & a_{i+1,i+2} & a_{i+1,i+3} & \cdots & a_{i+1,i+\ell+1} & \cdots \\ & & & \pi^{m_{i+2}} & a_{i+2,i+3} & & & \\ & & & & \ddots & \ddots & & \\ & & & & & \pi^{m_{i+\ell}} & a_{i+\ell,i+\ell+1} & \\ & & & & & & 0 & \pi^{m_{i+\ell+2}} \\ & & & & & & & \ddots \end{pmatrix}.$$

Recall that by induction, $a_{ij} = 0$ for $i+1 \le j \le i+\ell$. As a result, interchanging rows $i, i+1$, then $i+1, i+2, \ldots, i+\ell-1, i+\ell$ produces an upper triangular matrix with determinant $\frac{\det\xi}{\pi^{m_i+m_{i+\ell+1}}}a_{i,i+\ell+1}$ which because of the interchange of rows differs from the determinant of the minor by $(-1)^\ell$. It now follows that

$$C_{i,i+\ell+1} = (\pi^{m_i}d_i)(-1)\frac{a_{i,i+\ell+1}}{\pi^{m_i+m_{i+\ell+1}}} + \frac{a_{i,i+\ell+1}d_{i+\ell+1}}{\pi^{m_{i+\ell+1}}} = \frac{a_{i,i+\ell+1}}{\pi^{m_{i+\ell+1}}}(d_{i+\ell+1}-d_i).$$

As in the base case, since the d_k's are arbitrary elements of $\mathcal{O}$, ξ is in Hermite normal form, and we require $C_{i,i+\ell+1} \in \mathcal{O}$, it follows that $a_{i,i+\ell+1} = 0$, which completes the proof. □

Corollary 2.3. *Every maximal order in $M_n(k)$ containing a subring of the form $R = \operatorname{diag}(\mathcal{O},\ldots,\mathcal{O})$ has the form* $\Lambda(m_1,\ldots,m_n) = \begin{pmatrix} \mathcal{O} & \mathfrak{p}^{m_1-m_2} & \mathfrak{p}^{m_1-m_3} & \cdots & \mathfrak{p}^{m_1-m_n} \\ \mathfrak{p}^{m_2-m_1} & \mathcal{O} & \mathfrak{p}^{m_2-m_3} & \cdots & \mathfrak{p}^{m_2-m_n} \\ \mathfrak{p}^{m_3-m_1} & \mathfrak{p}^{m_3-m_2} & \ddots & \cdots & \mathfrak{p}^{m_3-m_n} \\ \vdots & \vdots & & \mathcal{O} & \vdots \\ \mathfrak{p}^{m_n-m_1} & \cdots & & \mathfrak{p}^{m_n-m_{n-1}} & \mathcal{O} \end{pmatrix}$.

In particular, $\Lambda(m_1,\ldots,m_n) = \Lambda(0, m_2-m_1,\ldots,m_n-m_1)$ is the order characterized by $E^{(i,i)}\Lambda(m_1,\ldots,m_n)E^{(j,j)} = \mathfrak{p}^{m_i-m_j}E^{(i,j)}$.

Proof. In Proposition 2.2, we observed that the maximal orders containing R all have the form $\xi^{-1}M_n(\mathcal{O})\xi$ where ξ is diagonal. For later convenience in identifying vertices with homothety classes of lattices below, we assume that ξ has the form $\xi = \mathrm{diag}(\pi^{-m_1}, \dots, \pi^{-m_n})$. Thus $\xi^{-1}M_n(\mathcal{O})\xi$ is certainly contained in the set $\Lambda(m_1, \dots, m_n)$. On the other hand, from Proposition 2.1, it is easily seen that the ij-entry of $\xi^{-1}M_n(\mathcal{O})\xi$ is an ideal containing $\pi^{m_i - m_j}$, which completes the proof. $\square$

2.3 Connections to the affine building for $SL_n(k)$

To introduce the connection between split orders in B and convex polytopes in affine buildings requires a bit of background which we present here in abbreviated form; the books by Brown [2] and Garrett [3] are two excellent resources for further details. Classically, affine buildings are associated to p-adic groups, e.g., $SL_n(k)$, and are characterized as simplicial complexes whose simplicial structure is determined by subgroups and cosets of the p-adic group being studied. Here, we give a well-known but more arithmetic characterization. To present the standard nomenclature, the simplicial complex which is the building is itself the union of subcomplexes called apartments, all of which are isomorphic. Apartments of an affine building are tilings of Euclidean space, and the structure of the tiling is determined by the associated Coxeter diagram which encodes the generators and relations of the Weyl group associated to the p-adic group.

The affine building for $SL_n(k)$ is an $(n-1)$-dimensional simplicial complex in which the maximal orders in $B = M_n(k)$ comprise the vertices. Apartments in the building are $(n-1)$-complexes, whose structure is captured by a tessellation of $\mathbb{R}^{n-1}$. We give a concrete realization; see [2] or [3] for further details. Let V be an n-dimensional vector space over the local field k, and identify $B = M_n(k)$ with $End_k(V)$. Let L be any lattice (free $\mathcal{O}$-module of rank n) in V. The homothety class of L, denoted $[L]$, is simply the set of lattices $\{\lambda L \mid \lambda \in k^\times\}$.

It is easy to check that for two lattices L and M, the homothety classes $[L] = [M]$ iff $End_{\mathcal{O}}(L) = End_{\mathcal{O}}(M)$, and that as L runs through the set of lattices of V, $End_{\mathcal{O}}(L)$ runs through the set of maximal orders of B. Thus, the vertices of our building originally given by maximal orders in B, may instead be identified with the homothety classes of lattices in V. To introduce the simplicial structure, we define the notion of incidence: we say that two vertices are incident if there are lattices L and L' representing the vertices such that $\pi L \subseteq L' \subseteq L$. Note in this case, $\pi L' \subseteq \pi L \subseteq L'$, so the definition of incidence is symmetric, and defines the edges (1-simplicies) in the building. An m-simplex is characterized by lattices L_i (representing its vertices) satisfying $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_m \subsetneq L_0$, or equivalently flags of length m in the $\mathcal{O}/\pi\mathcal{O}$-vector space $L_0/\pi L_0$. The maximal simplicies ($(n-1)$-simplicies) are called the chambers of the building.

To make things even more concrete, we note ([3]) that there is a one-to-one correspondence between sets of n linearly independent lines in V (frames) and apartments in the building for $SL_n(k)$. In particular, every vertex in a fixed apartment can be represented by a lattice of the form $\mathcal{O}\pi^{\nu_1}e_1 \oplus \cdots \oplus \mathcal{O}\pi^{\nu_n}e_n$ for some fixed basis $\{e_1, \dots, e_n\}$ of V, and where the ν_i range over all elements of $\mathbb{Z}$. Since each vertex in the apartment is the homothety class of a lattice $\mathcal{O}\pi^{\nu_1}e_1\oplus\cdots\oplus\mathcal{O}\pi^{\nu_n}e_n$, we may simply identify the vertices in an apartment in the $SL_n(k)$ building with the elements of $\mathbb{Z}^n/\mathbb{Z}(1,1,\ldots,1)$, where we represent the homothety class of $\mathcal{O}\pi^{\nu_1}e_1\oplus\cdots\oplus\mathcal{O}\pi^{\nu_n}e_n$ by $[\nu_1,\ldots,\nu_n]$ or after normalizing, by $[0,\nu_2-\nu_1,\ldots,\nu_n-\nu_1]$.

To recast some of our earlier algebraic results in this geometric setting, we let V be as above, fix a basis $\{e_1,\ldots,e_n\}$ for V and let L_0 be the $\mathcal{O}$-lattice with basis $\{e_i\}$. Identifying $\mathrm{End}_{\mathcal{O}}(L_0)$ with $\Lambda_0 = M_n(\mathcal{O})$, we observe that for $\xi\in B^\times$, $\xi^{-1}\Lambda_0\xi = \mathrm{End}(\xi^{-1}L_0)$, so all maximal orders in B have the form $\mathrm{End}(\xi^{-1}L_0)$ for some $\xi\in B^\times$. In Corollary 2.3, we showed that every maximal order containing $R=\mathrm{diag}(\mathcal{O},\ldots,\mathcal{O})$ can be expressed as $\Lambda(m_1,\ldots,m_n)=\Lambda(0,m_2-m_1,\ldots,m_n-m_1)$. So taking $\xi=\mathrm{diag}(1,\pi^{-m_2},\ldots,\pi^{-m_n})$, we can identify $\Lambda(0,m_2,\ldots,m_n)$ with the homothety class of the lattice $\xi^{-1}L_0=\mathcal{O}e_1\oplus\mathcal{O}\pi^{m_2}e_2\oplus\cdots\oplus\mathcal{O}\pi^{m_n}e_n$ which we denote $[0,m_2,\ldots,m_n]$. Thus the set of maximal orders containing R can be represented as vertices of the building given by homothety classes $[0,m_2,\ldots,m_n]$, $m_i\in\mathbb{Z}$.

Remark 2.4. The significance of the above characterization is twofold. First, every maximal order in this fixed apartment contains R, so that all such maximal orders are split orders. More significantly is that if we wish to consider orders S which contain R, the set of maximal orders which contain S all lie in a given apartment. Of course there may be many such apartments, but the ability to restrict to a fixed apartment leads not only to the concrete algebraic representation, but more importantly to the geometric one we develop below.

3 Geometric considerations

Our goal is to give a geometric characterization of split orders, and we begin in our restricted setting of split orders S of $B=M_n(k)$ with $R=\mathrm{diag}(\mathcal{O},\ldots,\mathcal{O})\subset S\subset B$. By Remark 2.4, we can and do fix an apartment $\mathcal{A}_0$ which contains all the maximal orders $\Lambda(0,m_2,\ldots,m_n)$ that contain a given S. Via a fixed basis for V (which yields the frame defining $\mathcal{A}_0$), we identify the apartment with $\mathbb{R}^{n-1}\cong\{0\}\times\mathbb{R}^{n-1}\subset\mathbb{R}^n$; the set of vertices in $\mathcal{A}_0$ is identified with $\{0\}\times\mathbb{Z}^{n-1}\cong\mathbb{Z}^n/\mathbb{Z}(1,\ldots,1)$. As noted after Proposition 2.1, we adopt the succinct presentation of S as $S=(\mathfrak{p}^{\nu_{ij}})=\begin{pmatrix}\mathcal{O} & & \mathfrak{p}^{\nu_{ij}}\\ & \ddots & \\ \mathfrak{p}^{\nu_{ij}} & & \mathcal{O}\end{pmatrix}$, with $\nu_{ij}\in\mathbb{Z}$, $\nu_{ii}=0$.

For a maximal order $\Lambda(0,m_2,\ldots,m_n)=\begin{pmatrix}\mathcal{O} & \mathfrak{p}^{-m_2} & \mathfrak{p}^{-m_3} & \ldots & \mathfrak{p}^{-m_n}\\ \mathfrak{p}^{m_2} & \mathcal{O} & \mathfrak{p}^{m_2-m_3} & \ldots & \mathfrak{p}^{m_2-m_n}\\ \mathfrak{p}^{m_3} & \mathfrak{p}^{m_3-m_2} & \ddots & \ldots & \mathfrak{p}^{m_3-m_n}\\ \vdots & \vdots & & \mathcal{O} & \vdots\\ \mathfrak{p}^{m_n} & \ldots & & \mathfrak{p}^{m_n-m_{n-1}} & \mathcal{O}\end{pmatrix}$,
we have $S\subset\Lambda(0,m_2,\ldots,m_n)$ if and only if (setting $m_1=0$)

$$-\nu_{ji}\le m_i-m_j\le\nu_{ij}\text{ for all }i,j. \tag{3.1}$$

Given our identification of the apartment $\mathcal{A}_0$ with $\mathbb{R}^{n-1}$, the equations of the form $L_{ij}:=x_i-x_j=\nu\in\mathbb{Z}$ are hyperplanes in $\mathbb{R}^{n-1}$ and represent a subset of the walls in the apartment;

they represent all of the walls if $n = 2, 3$. It is clear that the inequalities

$$-\nu_{ji} \le L_{ij} = x_i - x_j \le \nu_{ij} \tag{3.2}$$

define a convex polytope in $\mathbb{R}^{n-1}$ which we denote by C_S.

The immediate aim of this section is to establish a one-to-one correspondence between split orders containing R and convex polytopes of this form in the apartment $\mathcal{A}_0$. We have already seen (3.1) that the convex hull determined by the walls of the building which contain the set of maximal orders containing a given split order forms a convex polytope. We now further show that the split order is the intersection of the maximal orders contained in that polytope.

Definition 3.1. Given our fixed apartment $\mathcal{A}_0$, let $\mathcal{C}$ denote the set of convex polytopes determined by systems of inequalities as in (3.2); we denote a typical element in $\mathcal{C}$ as $C(\boldsymbol{\nu})$, $\boldsymbol{\nu} = (\nu_{ij}) \in M_n(\mathbb{Z})$. We shall require that $\boldsymbol{\nu}$ (or $C(\boldsymbol{\nu})$) be *reduced*, meaning the convex region determined by the inequalities (3.2) contain at least one vertex of the building, and each of the hyperplanes determined by the ν_{ij} meets the convex region. In the usual terminology of convex geometry, each of the given hyperplanes $L_{ij} = \nu_{ij}$ or $L_{ij} = -\nu_{ji}$ is a supporting hyperplane.

Remark 3.2. Note that since $x_1 = 0$ in our characterization of the apartment $\mathcal{A}_0$, the inequalities $-\nu_{1i} \le x_i - x_1 \le \nu_{i1}$ reduce to $-\nu_{1i} \le x_i \le \nu_{i1}$, so that $C(\boldsymbol{\nu})$ always defines a compact convex region, hence one containing only finitely many vertices.

Proposition 3.3. *Let $C = C(\boldsymbol{\nu}) \in \mathcal{C}$, and let $S_C = (\mathfrak{p}^{\mu_{ij}}) = \bigcap_{\Lambda\in C} \Lambda$ be the split order which is the intersection of all maximal orders in $C(\boldsymbol{\nu})$. Then $\mu_{ij} = \nu_{ij}$ for all i, j.*

Proof. Let Λ_k index the maximal orders (vertices) in $C(\boldsymbol{\nu})$, and denote $\Lambda_k = (\mathfrak{p}^{\lambda_{ij}^{(k)}})$. Since $S_C = (\mathfrak{p}^{\mu_{ij}})$ is the intersection of the Λ_k, it is clear that $\mu_{ij} = \max_k\{\lambda_{ij}^{(k)}\}$, so $\mu_{ii} = \lambda_{ii}^{(k)} = 0$. For each $i < j$ we have $-\nu_{ji} \le \lambda_{ij}^{(k)} \le \nu_{ij}$, so $\mu_{ij} = \max_k\{\lambda_{ij}^{(k)}\}$ and $\mu_{ji} = \max_k\{\lambda_{ji}^{(k)}\} = \max_k\{-\lambda_{ij}^{(k)}\} = -\min_k\{\lambda_{ij}^{(k)}\}$. However, since for the convex region $C(\boldsymbol{\nu})$, we require that $\boldsymbol{\nu}$ be reduced, there are maximal orders on the boundary of the region achieving each of the bounding limits. Thus for $i < j$, $\mu_{ij} = \max_k\{\lambda_{ij}^{(k)}\} = \nu_{ij}$, while $\mu_{ji} = -\min_k\{\lambda_{ij}^{(k)}\} = \nu_{ji}$. $\square$

Let's examine the correspondence as it now stands. Given $C = C(\boldsymbol{\nu}) \in \mathcal{C}$, we form $S_C = (\mathfrak{p}^{\mu_{ij}}) = \bigcap_{\Lambda\in C} \Lambda$, and since $\mu_{ij} = \nu_{ij}$, we have $C(\boldsymbol{\mu}) = C(\boldsymbol{\nu})$ which is half of the desired correspondence between split orders and convex polytopes. Perhaps more succinctly we have:

$$C = C(\boldsymbol{\nu}) \mapsto S_C = (\mathfrak{p}^{\mu_{ij}}) = \bigcap_{\Lambda\in C} \Lambda \mapsto C(\boldsymbol{\mu}) = C.$$

To establish the other half of the correspondence,

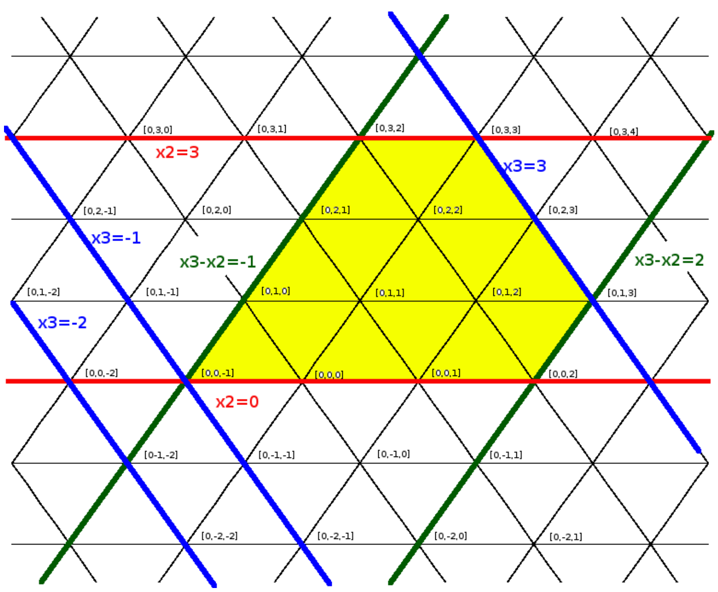
$$S = (\mathfrak{p}^{\nu_{ij}}) \mapsto C(\boldsymbol{\nu}) \mapsto \bigcap_{\Lambda\in C(\boldsymbol{\nu})} \Lambda = (\mathfrak{p}^{\mu_{ij}}) = S,$$

significantly more effort is required. Consider a subset of $M_n(k)$ having the form $S = (\mathfrak{p}^{\nu_{ij}})$. A necessary condition that S be contained in some maximal order is that $\nu_{ij} + \nu_{ji} \geq 0$ for all i, j. Given that necessary condition, S determines a convex polytope $C_S = C(\boldsymbol{\nu})$ via the pairs of inequalities in (3.2). The potential difficulty is that different subsets S can determine the same convex region. The following example demonstrates the difficulty and suggests its resolution.

Example 3.4. Consider $S = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathfrak{p} \\ \mathfrak{p}^3 & \mathcal{O} & \mathfrak{p} \\ \mathfrak{p}^3 & \mathfrak{p}^2 & \mathcal{O} \end{pmatrix}$ and $S' = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathfrak{p}^2 \\ \mathfrak{p}^3 & \mathcal{O} & \mathfrak{p} \\ \mathfrak{p}^3 & \mathfrak{p}^2 & \mathcal{O} \end{pmatrix}$. The diagram below (points have coordinates $[0, x_2, x_3]$) illustrates that S and S' determine the same convex region via the inequalities (3.2):

$$S: \quad \begin{aligned} 0 &\leq x_2 \leq 3 \\ -1 &\leq x_3 \leq 3 \\ -1 &\leq x_3 - x_2 \leq 2 \end{aligned} \qquad\qquad S': \quad \begin{aligned} 0 &\leq x_2 \leq 3 \\ -2 &\leq x_3 \leq 3 \\ -1 &\leq x_3 - x_2 \leq 2 \end{aligned}$$

From the diagram, we see that the hyperplane $x_3 = -2$ does not intersect the convex polytope, while the hyperplane $x_3 = -1$ does so in precisely one point, though neither is actually required to determine the convex region.



S is an order since it is the intersection of the maximal orders in this convex region, however S' is not. Indeed, S is the intersection of those maximal orders on the boundary of the convex polytope which determine it:

$$\begin{aligned} S &= \Lambda(0,0,-1) \cap \Lambda(0,3,2) \cap \Lambda(0,3,3) \cap \Lambda(0,1,3) \cap \Lambda(0,0,2) \\ &= \Lambda(0,0,-1) \cap \Lambda(0,3,3) \cap \Lambda(0,0,2) \\ &= \Lambda(0,0,-1) \cap \Lambda(0,3,2) \cap \Lambda(0,1,3) \\ &= \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathfrak{p} \\ \mathcal{O} & \mathcal{O} & \mathfrak{p} \\ \mathfrak{p}^{-1} & \mathfrak{p}^{-1} & \mathcal{O} \end{pmatrix} \cap \begin{pmatrix} \mathcal{O} & \mathfrak{p}^{-3} & \mathfrak{p}^{-2} \\ \mathfrak{p}^{3} & \mathcal{O} & \mathfrak{p} \\ \mathfrak{p}^{2} & \mathfrak{p}^{-1} & \mathcal{O} \end{pmatrix} \cap \begin{pmatrix} \mathcal{O} & \mathfrak{p}^{-1} & \mathfrak{p}^{-3} \\ \mathfrak{p} & \mathcal{O} & \mathfrak{p}^{-2} \\ \mathfrak{p}^{3} & \mathfrak{p}^{2} & \mathcal{O} \end{pmatrix} \end{aligned}$$

On the other hand it is easy to see that S' is not an order. By Proposition 2.1, a necessary condition that a subset $S = (\mathfrak{p}^{\nu_{ij}}) \supset R$ be an order is that it be closed under multiplication, which requires $\nu_{ik} + \nu_{kj} \geq \nu_{ij}$, $\nu_{kk} = 0$ for all i, j, k. We note that in S', $\nu_{12} + \nu_{23} = 0 + 1 \not\geq 2 = \nu_{13}$.

The key to establishing the other half of the desired correspondence is to connect the failure to be an order with a geometric condition. This leads us to the the following definition.

Definition 3.5. Let $S = (\mathfrak{p}^{\nu_{ij}})$ be a subset of $M_n(k)$ which contains R and satisfies $\nu_{ij}+\nu_{ji} \geq 0$ for all i, j. Call S *reduced* if the convex region it determines, $C(\boldsymbol{\nu})$, is reduced.

Proposition 3.6. *Let $S = (\mathfrak{p}^{\nu_{ij}})$ be as above. Then S is an order if and only if S is reduced.*

Remark 3.7. Note that if S is not reduced, there is an $\overline{S} \supset S$ which is reduced (hence an order), and which determines exactly the same convex polytope.

Proof. One direction is quite easy. If S is reduced, the bounds on the inequalities defining the convex polytope (3.2) are sharp, and from the arguments above, the intersection of all the maximal orders in that convex polytope equals S, that is S is the intersection of the maximal orders containing it, hence S is an order.

Note that by Proposition 2.1, $S = (\mathfrak{p}^{\nu_{ij}})$ is an order if and only if $\nu_{ik} + \nu_{kj} \geq \nu_{ij}$ for every i, j, k. So to establish the converse of our theorem, we show that $\nu_{ik} + \nu_{kj} \geq \nu_{ij}$ for every i, j, k implies $\boldsymbol{\nu} = (\nu_{ij})$ (i.e., $C(\boldsymbol{\nu})$) is reduced. We proceed by contradiction, so we assume that there exist i_0, j_0 such that $x_{i_0} - x_{j_0} = \nu_{i_0 j_0}$ does not intersect $C(\boldsymbol{\nu})$.

Since $x_1 = 0$, there is some asymmetry in the expression $x_{i_0} - x_{j_0}$ when one of $i_0, j_0 = 1$, so we separate the proof into cases beginning with the generic case.

Case: $\boldsymbol{i_0, j_0 \neq 1}$. If the hyperplane $x_{i_0} - x_{j_0} = \nu_{i_0 j_0}$ does not intersect $C(\boldsymbol{\nu})$, we have $x_{i_0} - x_{j_0} < \nu_{i_0 j_0}$ for all $(x_i) \in C(\boldsymbol{\nu})$. Note that the symmetric case $x_{i_0} - x_{j_0} > -\nu_{j_0 i_0}$ is equivalent to $x_{j_0} - x_{i_0} < \nu_{j_0 i_0}$ so we consider only $x_{i_0} - x_{j_0} < \nu_{i_0 j_0}$. Let $\boldsymbol{b} = (b_i) \in C(\boldsymbol{\nu})$ achieve a maximum for $x_{i_0} - x_{j_0}$, say $b_{i_0} - b_{j_0} = \mu_{i_0 j_0} < \nu_{i_0 j_0}$. To arrive at the desired contradiction, we use $\boldsymbol{b}$ to construct a point $\boldsymbol{b}' = (b'_i) \in C(\boldsymbol{\nu})$ with $\mu_{i_0 j_0} < b'_{i_0} - b'_{j_0} \leq \nu_{i_0 j_0}$. Note, throughout the proof we use without further mention that all hyperplanes have the form $x_i - x_j = \nu \in \mathbb{Z}$.

We set a bit of notation. Let $\boldsymbol{e}_\ell$ be the ℓth standard basis vector in $\mathbb{R}^n$, and for $k \neq 1, i_0, j_0$, let

$$\alpha_k = \begin{cases} 1 & \text{if } b_k - b_{j_0} = \nu_{k j_0}, \\ 0 & \text{otherwise,} \end{cases} \qquad \beta_k = \begin{cases} 1 & \text{if } b_{i_0} - b_k = \nu_{i_0 k}, \\ 0 & \text{otherwise.} \end{cases}$$

To define $\boldsymbol{b}'$ we need to increase the difference $b_{i_0} - b_{j_0}$, either by increasing b_{i_0} or decreasing b_{j_0} and adjust the other coordinates to satisfy all the remaining convexity bounds. We put

$$\boldsymbol{b}' = \begin{cases} \boldsymbol{b} - \boldsymbol{e}_{j_0} - \sum_{k\neq 1,i_0,j_0} \alpha_k \boldsymbol{e}_k & \text{if } b_{i_0} = \nu_{i_0 1}, \\ \boldsymbol{b} + \boldsymbol{e}_{i_0} + \sum_{k\neq 1,i_0,j_0} \beta_k \boldsymbol{e}_k & \text{if } b_{i_0} < \nu_{i_0 1}. \end{cases} \tag{3.3}$$

Subcase A. We begin with the case where $b_{i_0} = \nu_{i_0 1}$ and $\boldsymbol{b}' = \boldsymbol{b} - \boldsymbol{e}_{j_0} - \sum_{k\neq 1,i_0,j_0} \alpha_k \boldsymbol{e}_k$.

First we show that $-\nu_{1i} \leq b'_i = b'_i - b'_1 \leq \nu_{i1}$ for all i. This is clear for $b'_1 = 0$ and $b'_{i_0} = b_{i_0} = \nu_{i_0 1}$. We note $b'_{j_0} = b_{j_0} - 1 \leq \nu_{j_0 1} - 1 \leq \nu_{j_0 1}$. To see $b'_{j_0} \geq -\nu_{1j_0}$, note that $b_{i_0} - b_{j_0} = \mu_{i_0 j_0} < \nu_{i_0 j_0} \leq \nu_{i_0 1} + \nu_{1 j_0}$ by assumptions on $S = (\mathfrak{p}^{\nu_{ij}})$ and $\boldsymbol{b}$, so

$$b_{j_0} = b_{i_0} - \mu_{i_0 j_0} = \nu_{i_0 1} - \mu_{i_0 j_0} > \nu_{i_0 1} - \nu_{i_0 j_0} \geq -\nu_{1 j_0}. \tag{3.4}$$

Thus $b_{j_0} > -\nu_{1j_0}$ implies $b'_{j_0} = b_{j_0} - 1 \geq -\nu_{1j_0}$ as desired. Next we finish the remaining inequalities of the form $-\nu_{1k} \leq b'_k = b'_k - b'_1 \leq \nu_{k1}$, $k \neq 1, i_0, j_0$.

By the definition of $\boldsymbol{b}'$, if $b_k - b_{j_0} < \nu_{kj_0}$, then $b'_k = b_k$, so there is no issue. If $b_k - b_{j_0} = \nu_{kj_0}$, then $b'_k = b_k - 1$, so of course $b'_k \leq \nu_{k1}$. To see $b'_k \geq -\nu_{1k}$, we suppose not, so $b'_k = b_k - 1 < -\nu_{1k}$, hence $b_k < -\nu_{1k} + 1$. On the other hand, $\boldsymbol{b} \in C(\boldsymbol{\nu})$ implies $b_k \geq -\nu_{1k}$ from which we deduce $b_k = -\nu_{1k}$. Now $b_k - b_{j_0} = \nu_{kj_0}$ implies $b_{j_0} = b_k - \nu_{kj_0} = -\nu_{1k} - \nu_{kj_0} \leq -\nu_{1j_0}$ contrary to equation (3.4).

Next we must consider bounds on $b'_k - b'_\ell$ where $k, \ell \neq 1$, and where $\{k, \ell\} \cap \{i_0, j_0\}$ has cardinality 0, 1, or 2.

First observe that since $\mu_{i_0 j_0} < \nu_{i_0 j_0}$ (and both are integers),

$$-\nu_{j_0 i_0} \leq b_{i_0} - b_{j_0} < b'_{i_0} - b'_{j_0} = b_{i_0} - (b_{j_0} - 1) = \mu_{i_0 j_0} + 1 \leq \nu_{i_0 j_0}$$

Next consider

$$-\nu_{j_0 k} \leq b_k - b_{j_0} \leq b'_k - b'_{j_0} = \begin{cases} b_k - b_{j_0} + 1 & \text{if } b_k - b_{j_0} < \nu_{kj_0}, \\ b_k - b_{j_0} & \text{if } b_k - b_{j_0} = \nu_{kj_0} \end{cases} \leq \nu_{kj_0}.$$

Similarly, since

$$b'_{i_0} - b'_k = \begin{cases} b_{i_0} - b_k + 1 & \text{if } b_k - b_{j_0} = \nu_{kj_0}, \\ b_{i_0} - b_k & \text{if } b_k - b_{j_0} < \nu_{kj_0}, \end{cases}$$

it is clear that $-\nu_{ki_0} \leq b'_{i_0} - b'_k$, and the only potential issue with the upper bound is when $b_k - b_{j_0} = \nu_{kj_0}$. If indeed $b'_{i_0} - b'_k > \nu_{i_0 k}$, then $b_{i_0} - b_k > \nu_{i_0 k} - 1$ which means $b_{i_0} - b_k = \nu_{i_0 k}$. But this together with $b_k - b_{j_0} = \nu_{kj_0}$ implies $b_{i_0} - b_{j_0} = \nu_{i_0 k} + \nu_{kj_0} \geq \nu_{i_0 j_0}$, but by hypothesis $b_{i_0} - b_{j_0} = \mu_{i_0 j_0} < \nu_{i_0 j_0}$, a contradiction.

Finally, we come to the case $b'_k - b'_\ell$ where $k, \ell \neq 1$, and where $\{k, \ell\} \cap \{i_0, j_0\} = \emptyset$. We see that

$$b'_k - b'_\ell = \begin{cases} b_k - b_\ell + 1 & \text{if } b_k - b_{j_0} < \nu_{kj_0} \text{ and } b_\ell - b_{j_0} = \nu_{\ell j_0}, \\ b_k - b_\ell - 1 & \text{if } b_k - b_{j_0} = \nu_{kj_0} \text{ and } b_\ell - b_{j_0} < \nu_{\ell j_0}, \\ b_k - b_\ell & \text{otherwise.} \end{cases}$$

Everything is clear except for the upper bound in the first case and the lower bound in the second case. Assuming $b_k - b_{j_0} < \nu_{kj_0}$ and $b_\ell - b_{j_0} = \nu_{\ell j_0}$, if $b_k - b_\ell + 1 > \nu_{k\ell}$, then $b_k - b_\ell = \nu_{k\ell}$. This implies $b_k - b_{j_0} = \nu_{k\ell} + \nu_{\ell j_0} \geq \nu_{kj_0}$, a contradiction. Analogously, assuming $b_k - b_{j_0} = \nu_{kj_0}$ and $b_\ell - b_{j_0} < \nu_{\ell j_0}$, if $b_k - b_\ell - 1 < -\nu_{\ell k}$ then $b_k - b_\ell = -\nu_{\ell k}$ which $b_\ell - b_{j_0} = \nu_{\ell k} + \nu_{kj_0} \geq \nu_{\ell j_0}$, a contradiction.

Subcase B. Here we assume $b_{i_0} < \nu_{i_0 1}$ and $\boldsymbol{b}' = \boldsymbol{b} + \boldsymbol{e}_{i_0} + \sum_{k \neq 1, i_0, j_0} \beta_k \boldsymbol{e}_k$,

$$\beta_k = \begin{cases} 1 & \text{if } b_{i_0} - b_k = \nu_{i_0 k}, \\ 0 & \text{otherwise.} \end{cases}$$

First we show that $-\nu_{1i} \leq b'_i = b'_i - b'_1 \leq \nu_{i1}$ for all i. This is clear for $b'_1 = 0$ and $b'_{j_0} = b_{j_0}$. We note $-\nu_{1i_0} \leq b_{i_0} < b'_{i_0} = b_{i_0} + 1 \leq \nu_{i_0 1}$ since $b_{i_0} < \nu_{i_0 1}$. For $k \neq 1, i_0, j_0$, the only issue is when $b_{i_0} - b_k = \nu_{i_0 k}$ in which case $b'_k = b_k + 1$, and then only concerns the upper bound. If $b'_k > \nu_{k1}$, then $b_k = \nu_{k1}$, so that $b_{i_0} = b_k + \nu_{i_0 k} = \nu_{i_0 k} + \nu_{k1} \geq \nu_{i_0 1}$, contrary to assumption.

Next observe

$$-\nu_{j_0 i_0} \leq b_{i_0} - b_{j_0} < b'_{i_0} - b'_{j_0} = b_{i_0} - b_{j_0} + 1 = \mu_{i_0 j_0} + 1 \leq \nu_{i_0 j_0}.$$

We also have

$$-\nu_{k i_0} \leq b'_{i_0} - b'_k = \begin{cases} b_{i_0} - b_k & \text{if } b_{i_0} - b_k = \nu_{i_0 k}, \\ b_{i_0} - b_k + 1 & \text{if } b_{i_0} - b_k < \nu_{i_0 k} \end{cases} \leq \nu_{i_0 k}.$$

Similarly, since

$$b'_k - b'_{j_0} = \begin{cases} b_k - b_{j_0} & \text{if } b_{i_0} - b_k < \nu_{i_0 k}, \\ b_k - b_{j_0} + 1 & \text{if } b_{i_0} - b_k = \nu_{i_0 k}, \end{cases}$$

the only issue is with the upper bound when $b_{i_0} - b_k = \nu_{i_0 k}$. If $b'_k - b'_{j_0} > \nu_{kj_0}$, then $b_k - b_{j_0} = \nu_{kj_0}$. This together with $b_{i_0} - b_k = \nu_{i_0 k}$ implies $b_{i_0} - b_{j_0} = \nu_{i_0 k} + \nu_{kj_0} \geq \nu_{i_0 j_0}$, contrary to our original assumption.

Finally, we come to the case $b'_k - b'_\ell$ where $k, \ell \neq 1$, and where $\{k, \ell\} \cap \{i_0, j_0\} = \emptyset$. We see that

$$b'_k - b'_\ell = \begin{cases} b_k - b_\ell + 1 & \text{if } b_{i_0} - b_k = \nu_{i_0 k} \text{ and } b_{i_0} - b_\ell < \nu_{i_0 \ell}, \\ b_k - b_\ell - 1 & \text{if } b_{i_0} - b_k < \nu_{i_0 k} \text{ and } b_{i_0} - b_\ell = \nu_{i_0 \ell}, \\ b_k - b_\ell & \text{otherwise.} \end{cases}$$

Everything is clear except for the upper bound in the first case and the lower bound in the second case. Assuming $b_{i_0} - b_k = \nu_{i_0 k}$ and $b_{i_0} - b_\ell < \nu_{i_0 \ell}$, if $b_k - b_\ell + 1 > \nu_{k\ell}$, then $b_k - b_\ell = \nu_{k\ell}$, so that $b_{i_0} - b_\ell = \nu_{i_0 k} + \nu_{k\ell} \geq \nu_{i_0 \ell}$, contrary to assumption. Analogously, assuming $b_{i_0} - b_k < \nu_{i_0 k}$ and $b_{i_0} - b_\ell = \nu_{i_0 \ell}$, if $b_k - b_\ell - 1 < \nu_{\ell k}$, then $b_k - b_\ell = -\nu_{\ell k}$, so $b_{i_0} - b_k = \nu_{i_0 \ell} + \nu_{\ell k} \geq \nu_{i_0 k}$, contrary to assumption.

Case: $\boldsymbol{i_0}$ or $\boldsymbol{j_0 = 1}$. We choose $\boldsymbol{b}$ as in the first case and define α_k and β_k exactly as before (noting the obvious redundant conditions on k). We put

$$\boldsymbol{b}' = \begin{cases} \boldsymbol{b} - \boldsymbol{e}_{j_0} - \sum_{k \neq 1, i_0, j_0} \alpha_k \boldsymbol{e}_k & \text{if } i_0 = 1, \\ \boldsymbol{b} + \boldsymbol{e}_{i_0} + \sum_{k \neq 1, i_0, j_0} \beta_k \boldsymbol{e}_k & \text{if } j_0 = 1. \end{cases} \tag{3.5}$$

Then these boundary cases are handled in exactly the same way as above with no further insights required, and this completes the proof of the theorem. □

We summarize both pieces of the correspondence as

Theorem 3.8. *There is a one-to-one correspondence between convex polytopes in $\mathcal{C}$ determined by the walls of the apartment $\mathcal{A}_0$ in the building for $SL_n(k)$ and split orders in $M_n(k)$ which contain R. The maps $C = C(\boldsymbol{\nu}) \mapsto S_C = (\mathfrak{p}^{\mu_{ij}}) = \bigcap_{\Lambda \in C} \Lambda$ and $S = (\mathfrak{p}^{\nu_{ij}}) \mapsto C(\boldsymbol{\nu})$ are inverse to one another.*

Proof. Given $C(\boldsymbol{\nu}) \in \mathcal{C}$, we have $\boldsymbol{\nu}$ is reduced, so by Proposition 3.3, $C(\boldsymbol{\nu}) \mapsto S_C = \bigcap_{\Lambda \in C} \Lambda = (\mathfrak{p}^{\nu_{ij}}) \mapsto C(\boldsymbol{\nu})$. On the other hand if $S = (\mathfrak{p}^{\mu_{ij}})$ is a split order, then by Proposition 3.6, $\boldsymbol{\mu} = (\mu_{ij})$ is reduced, so that $S = (\mathfrak{p}^{\mu_{ij}}) \mapsto C(\boldsymbol{\mu}) \mapsto \bigcap_{\Lambda \in C(\boldsymbol{\mu})} \Lambda = (\mathfrak{p}^{\mu_{ij}})$ by Proposition 3.3. □

Now we generalize the above results to that of our general notion of a split order. We begin by showing the intersection of any finite collection of maximal orders in a fixed apartment is a split order.

Proposition 3.9. *Let $\mathcal{A}$ be any apartment in the affine building for $SL_n(k)$, and let $\Lambda_1, \dots, \Lambda_r$ be maximal orders in $M_n(k)$ corresponding to vertices in $\mathcal{A}$. Then $S = \bigcap_{i=1}^r \Lambda_i$ is a split order.*

Proof. Our original fixed apartment $\mathcal{A}_0$ corresponds to the basis $\{e_i\}$ of the vector space V. Let $\{f_i\}$ be a basis of V whose frame determines the apartment $\mathcal{A}$. Let $\gamma \in GL_n(k)$ be the change of basis matrix taking e_i to f_i. Each maximal order $\Lambda_k = \mathrm{End}_{\mathcal{O}}(L_k)$ for a lattice $L_k = \oplus \mathcal{O}\pi^{a_i^{(k)}} f_i$. Let $\tilde{L}_k = \gamma^{-1} L_k = \oplus \mathcal{O}\pi^{a_i^{(k)}} e_i$ and $\tilde{\Lambda}_k = \mathrm{End}_{\mathcal{O}}(\tilde{L}_k)$. Then

$$\Lambda_k = \mathrm{End}_{\mathcal{O}}(L_k) = \mathrm{End}(\gamma \tilde{L}_k) = \gamma \mathrm{End}_{\mathcal{O}}(\tilde{L}_k)\gamma^{-1} = \gamma \tilde{\Lambda}_k \gamma^{-1}.$$

Now all of the $\tilde{\Lambda}_k$ are maximal orders in $\mathcal{A}_0$, which by Remark 2.4 all contain R. Thus, $S = \bigcap_{i=1}^r \Lambda_i \supset \gamma R \gamma^{-1}$, hence is a split order. □

Next we consider the converse.

Proposition 3.10. *Suppose that S is an order of $B = M_n(k)$ which contains $\gamma R \gamma^{-1}$ for some $\gamma \in B^\times$. Then S is the intersection of maximal orders lying in a convex polytope in the apartment $\mathcal{A} = \gamma \mathcal{A}_0$.*

Proof. If $S \supset \gamma R \gamma^{-1}$, then $\gamma^{-1} S \gamma$ is an order of B containing R. By Propositions 3.3 and 3.6, $\gamma^{-1} S \gamma = (\mathfrak{p}^{\boldsymbol{\nu}}) = \bigcap_{\tilde{\Lambda} \in C(\boldsymbol{\nu})} \tilde{\Lambda}$, that is $\boldsymbol{\nu}$ is reduced and $\gamma^{-1} S \gamma$ is the intersection of all the maximal orders $\tilde{\Lambda}$ in the convex polytope $C(\boldsymbol{\nu})$. It follows that

$$S = \gamma \left(\bigcap_{\tilde{\Lambda} \in C(\boldsymbol{\nu})} \tilde{\Lambda} \right) \gamma^{-1} = \bigcap_{\tilde{\Lambda} \in C(\boldsymbol{\nu})} \gamma \tilde{\Lambda} \gamma^{-1}.$$

Now let $\tilde{\Lambda} = \mathrm{End}_{\mathcal{O}}(\tilde{L})$ and $\tilde{\Lambda}' = \mathrm{End}_{\mathcal{O}}(\tilde{L}')$ be two maximal orders in $\mathcal{C}(\boldsymbol{\nu})$. Then $\gamma\tilde{\Lambda}\gamma^{-1} = \mathrm{End}_{\mathcal{O}}(\gamma\tilde{L})$ and $\gamma\tilde{\Lambda}'\gamma^{-1} = \mathrm{End}_{\mathcal{O}}(\gamma\tilde{L}')$. Since γ can simply be viewed as a change of basis matrix, the elementary divisors of L' in L, denoted $\{L : L'\}$, equal those of $\gamma L'$ in γL, that is $\{L : L'\} = \{\gamma L : \gamma L'\}$. Moreover, since the incidence relations among vertices in the building are determined by chains of lattices whose relative containments in an apartment are completely determined by the elementary divisors, we see that the collection of maximal orders (vertices) $\gamma\tilde{\Lambda}\gamma^{-1}$ have the same geometric configuration as do the collection of $\tilde{\Lambda} \in \mathcal{C}(\boldsymbol{\nu})$, that is, they form a convex polytope in the apartment $\mathcal{A} = \gamma\mathcal{A}_0$. $\square$

Finally, via propositions 3.9 and 3.10 we summarize the correspondence between general split orders and convex polytopes in the building as our main theorem.

Theorem 3.11. *There is a one-to-one correspondence between convex polytopes (as described by Equation 3.2) in apartments of the affine building for $SL_n(k)$ and split orders in $B = M_n(k)$.*

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