8. Spectral results on $N(j)$

Definition. Two linear maps $j, j' : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ are called isospectral if for each $Z \in \mathfrak{z}$, the maps $j_Z, j'_Z \in \mathfrak{so}(\mathfrak{v})$ have the same eigenvalues (with multiplicities) in $\mathbb{C}$.

Proposition [GGSWW]

Let $j, j' : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ be isospectral, and let $\mathcal{L}$ be a cocompact lattice in $\mathfrak{z}$ $\Rightarrow$ the associated closed Riemannian manifolds $N(j)$ and $N(j')$ are isospectral for the Laplace operator on functions.

Definition.

$j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ is of Heisenberg type if $j_Z^2 = -|Z|^2 \mathrm{Id}_{\mathfrak{v}}$ for all $Z \in \mathfrak{z}$.

Remark: If $j, j' : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ are both of Heisenberg type $\Rightarrow j$ and j' are obviously isospectral because the eigenvalues for both of them are $\pm i|Z|$, each with multiplicity $(\dim \mathfrak{v})/2$.

9. The special family $N^{a,b} := N(j^{a,b})$

Notation: $\mathbb{H} = \mathrm{span}\{1, i, j, k\}$ denote the algebra of quaternions with the usual multiplication, endowed with the inner product for which $\{1, i, j, k\}$ is an orthonormal basis.

Definition. For $a, b \in \mathbb{N}_0$ with $a + b > 0$ define

- $\mathfrak{v} := \mathbb{H} \oplus \cdots^{a+b} \cdots \oplus \mathbb{H}$ (orthogonal sum),
- $\mathfrak{z} := \mathrm{span}\{i, j, k\}$, the space of pure quaternions,
- $\mathcal{L} := \mathrm{span}_{\mathbb{Z}}\{i, j, k\}$, the standard lattice in $\mathfrak{z}$.

$\Rightarrow$ Define $j^{a,b} : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ by

$$j_Z^{a,b}(X_1, \ldots, X_a, Y_1, \ldots, Y_b) := (X_1 Z, \ldots, X_a Z, ZY_1, \ldots, ZY_b).$$

$\Rightarrow$ We denote the resulting Riemannian manifolds by

$$N^{a,b} := N(j^{a,b}), \text{ resp. } \tilde{N}^{a,b} := \tilde{N}(j^{a,b}).$$

10. Szabó's isospectral pairs $N^{a+b,0}$ and $N^{a,b}$

Remark: For all pairs $(a, b) \in \mathbb{N}_0^2$ with fixed sum

$$a + b = (\dim \mathfrak{v})/4 > 0$$

the associated Riemannian manifolds $N^{a,b}$ are of Heisenberg type and thus mutually isospectral.

Proposition [Sz]

For every $a \in \mathbb{N}$ the manifolds $N^{a,0}$ and $\tilde{N}^{a,0}$ are homogeneous.

Remark [Sz]: $N^{a,b}$ is not locally homogeneous if both a and b are nonzero.

$\Rightarrow$ One can not hear the local homogeneity property of a closed Riemannian manifold.

11. Weak local symmetry of $N^{a,0}$

Weakly symmetric spaces were introduced by A. Selberg in 1956.

A Riemannian manifold M is called weakly symmetric if each $p \in M$ and each nontrivial geodesic γ starting in p there exists an isometry f of M which fixes p and reverses γ (equivalently: $df_p(\dot\gamma(0)) = -\dot\gamma(0)$).

This is not Selberg's original definition, but was Z.I. Szabó's definition of what he called ray symmetry in 1993.

Now, for any given point $p \in \tilde{N}^{a,0}$ and any given tangent vector at p, we find an isometry f of $\tilde{N}^{a,0}$ which fixes p and whose differential maps the given tangent vector to its negative. In particular, since $\tilde{N}^{a,0}$ is homogeneous we only consider the case $p := \exp^{j^{a,0}}((1, 0, \ldots, 0), 0)$. Therefore, we conclude

Theorem:

For any $a \in \mathbb{N}$ the Riemannian manifold $\tilde{N}^{a,0}$ is weakly symmetric.

$\Rightarrow$ Since $\tilde{N}^{a,0}$ and $N^{a,0}$ are locally isometric, the manifold $N^{a,0}$ is weakly locally symmetric.

Remark: The isometry f in the proof of the Theorem will in general not descend to the quotient manifold $N^{a,0}$. So we cannot conclude weak symmetry of $N^{a,0}$ but only weak local symmetry.

12. Failure of the type $\mathcal{A}$ condition for $N^{a,b}$ with $a, b > 0$

Notation:

(i) Let $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ be any linear map (not necessarily one of our maps $j^{a,b}$). Inner products $\langle\ ,\ \rangle$ and norms $|\,.\,|$ will refer to the metric $g(j)$ on $\tilde{N}(j)$. We denote the Levi Civita connection and the Ricci tensor of $\tilde{N}(j)$ by ∇ and by **ric**, resp.

(iv) We identify vectors in $T_pG(j) = L_{p*}\mathfrak{g}(j)$ with their preimage in $\mathfrak{g}(j)$. Correspondingly, we will decompose $Y \in T_pG(j)$ as

$$Y = Y^{\mathfrak{v}} + Y^{\mathfrak{z}} \text{ with } Y^{\mathfrak{v}} \in \mathfrak{v},\ Y^{\mathfrak{z}} \in \mathfrak{z}.$$

Lema: Let j be of Heisenberg type and let $p = \exp^j(x, z) \in \tilde{N}(j)$, where $x \in \mathfrak{v}$, $|x| = 1$, $z \in \mathfrak{z}$. Then for all $Y_1, Y_2, Y \in T_p\tilde{N}(j)$ we have

$$\text{(i) } \mathrm{ric}_p(Y_1, Y_2) = \left(\frac{1}{4}\dim \mathfrak{v} - \frac{1}{2}\right)\langle Y_1^{\mathfrak{z}}, Y_2^{\mathfrak{z}}\rangle + \frac{1}{2}\langle [Y_1^{\mathfrak{v}}, x]^j, [Y_2^{\mathfrak{v}}, x]^j\rangle$$

$$+ \left(\dim \mathfrak{v} - 2 - \frac{1}{2}\dim \mathfrak{z}\right)\langle Y_1^{\mathfrak{v}}, Y_2^{\mathfrak{v}}\rangle + \frac{1}{2}(\dim \mathfrak{v} - 2)\langle j_{Y_1^{\mathfrak{z}}} Y_2^{\mathfrak{v}} + j_{Y_2^{\mathfrak{z}}} Y_1^{\mathfrak{v}},\ x\rangle,$$

$$\text{(ii) } (\nabla_Y \mathrm{ric})(Y, Y) = \langle [Y^{\mathfrak{v}}, x]^j, [j_{Y^{\mathfrak{z}}} Y^{\mathfrak{v}}, x]^j\rangle.$$

Theorem:

For $a, b > 0$ the Riemannian manifolds $\tilde{N}^{a,b}$ are not of Type $\mathcal{A}$.

$\Rightarrow$ Since the type $\mathcal{A}$ condition is a local condition and since $N^{a,b}$ and $\tilde{N}^{a,b}$ are locally isometric, we conclude that $N^{a,b}$ are not of Type $\mathcal{A}$.