1. Preliminaries

Throughout our work,

- (M, g) will be a closed manifold; i.e., a compact, connected, n-dimensional smooth Riemannian manifold, without boundary.
- $\Delta = -\textbf{div grad}$ will be the Laplace operator associated with g, acting on functions.

Moreover, we will assume the knowledge of the spectrum of Δ and attempts to determine information about the geometry of the given (M, g). The following concepts are basic:

Definition. Two compact (M, g, Δ) and (M', g', Δ') are said to be isospectral if the spectrums of Δ and Δ' coincide.

Definition. A geometric property of a compact Riemannian manifold M can be heard if it can be determined from the spectrum of Δ.

Definition. A geometric property is inaudible, i.e. not determined by the spectrum of Δ, if there exist pairs of isospectral manifolds which differ with respect to this property.

2. Motivation

Some positive results:

- The coefficients of the asymtotic expansion of the heat kernel of M, namely heat invariants, can be heard: $a_0 = \int_M \omega = Vol(M, g)$,

$$a_1 = \frac{1}{6}\int_M \tau\omega, \; a_2 = \frac{1}{360}\int_M (2|\mathrm{R}|^2 - 2|\mathrm{ric}|^2 + 5\tau^2)\omega, \ldots$$

Some negative results:

- Gordon, Gornet, Schueth, Webb, Wilson, 1998 in [GGSWW]: The maximum of the scalar curvature τ is inaudible.
- Schueth, 1999 in [S]: $\int_M |\mathrm{ric}|^2$ is inaudible.

⇛ Note that each a_k is some combination of expressions obtained from the Riemannian curvature tensor of M by applying covariant derivatives and traces. Therefore, it is of particular interest to know which curvature properties of a Riemannian manifold are spectrally determined.

3. Symmetric-like manifolds

The least restrictive of the curvature properties which we are going to consider is the so-called type $\mathcal{A}$ property introduced by A. Gray in 1978 as one of the possible natural extensions of Einstein spaces.

A Riemannian manifold M is said to be of type $\mathcal{A}$ if its Ricci tensor ric is cyclic parallel; that is, if $(\nabla_X \mathrm{ric})(X, X) = 0$ for all $X \in TM$.

Besides Einstein spaces, the most well-known examples of of type $\mathcal{A}$ spaces are locally symmetric spaces (characterize by $\nabla R = 0$), D'Atri spaces, and $\mathfrak{C}$-spaces. D'Atri spaces were introduced by J.E. D'Atri and H.K. Nickerson in 1969.

A Riemannian manifold is called a D'Atri space if the local geodesic symmetries (defined as $\sigma_p : \exp_p(X) \mapsto \exp_p(-X)$ on normal neighborhoods of p) preserve the Riemannian volume.

$\mathfrak{C}$-spaces were introduced by J. Berndt and L. Vanhecke in 1992.

A Riemannian manifold is a $\mathfrak{C}$-space if for each geodesic γ in M the eigenvalues of the associated field of Jacobi operators $R_{\dot\gamma(t)} := R(\,.\,, \dot\gamma(t))\dot\gamma(t)$ are constant in t.

Probabilistic commutative spaces were introduced by P.H. Roberts and H.D. Ursell in 1960 for compact Riemannian manifolds from a probabilistic point of view. The general case has been treated by O. Kowalski and F. Prüfer in 1982 and 1989.

Probabilistic commutative spaces are those Riemannian manifolds for which all Euclidean Laplacians $\tilde{\Delta}^{(k)}$ $(k \in \mathbb{N})$ commute.

The Euclidean Laplacians are defined as follows: For $p \in M$ define a differential operator $\bar{\Delta}_p$ on a normal neighborhood of p as the pullback of the (standard) Laplacian on the euclidean space T_pM via the exponential map $\exp_p$. Then $(\tilde{\Delta}^{(k)} f)(p)$ is defined as $((\bar{\Delta}_p)^k f)(p)$ for $f \in C^\infty(M)$.

Note that the previous properties are *local* properties of Riemannian manifolds. In this context, let us introduce the following definition of weakly locally symmetric spaces:

A Riemannian manifold M is called weakly locally symmetric if for every $p \in M$ there exists $\varepsilon > 0$ such that for any unit speed geodesic γ in M with $\gamma(0) = p$ there exists an isometry of the distance ball $B_\varepsilon(p)$ which fixes p and reverses $\gamma|_{(-\varepsilon,\varepsilon)}$.

Finally, $\mathfrak{TC}$ and $\mathfrak{GC}$ properties are two other (less studied) local properties of Riemannian manifolds introduced by J. Berndt and L. Vanhecke in 1993 and, by the previous authors and F. Prüfer in 1995, respectively.

(M, g) is a $\mathfrak{TC}$-space if $\forall m \in M$ and $\forall p \in M$ sufficiently close to m, $T_p(m)$ and $T_{\sigma_m(p)}(m)$ have the same eigenvalues.

Here, $T_p(m)$ denotes the Shape operator at the point $m \in M$ of the geodesic sphere with center p and radius r.

(M, g) is a $\mathfrak{GC}$-space if $\forall m \in M$ and $\forall p \in M$ sufficiently close to m, $T_m(p)$ and $\sigma_{m^*}^{-1} \circ T_m(\sigma_m(p)) \circ \sigma_{m^*}$ have the same eigenvalues.