On inaudible curvature properties of closed Riemannian manifolds, *Ann. Glob. Anal. Geom.* **37** *(2010), 339–349.*

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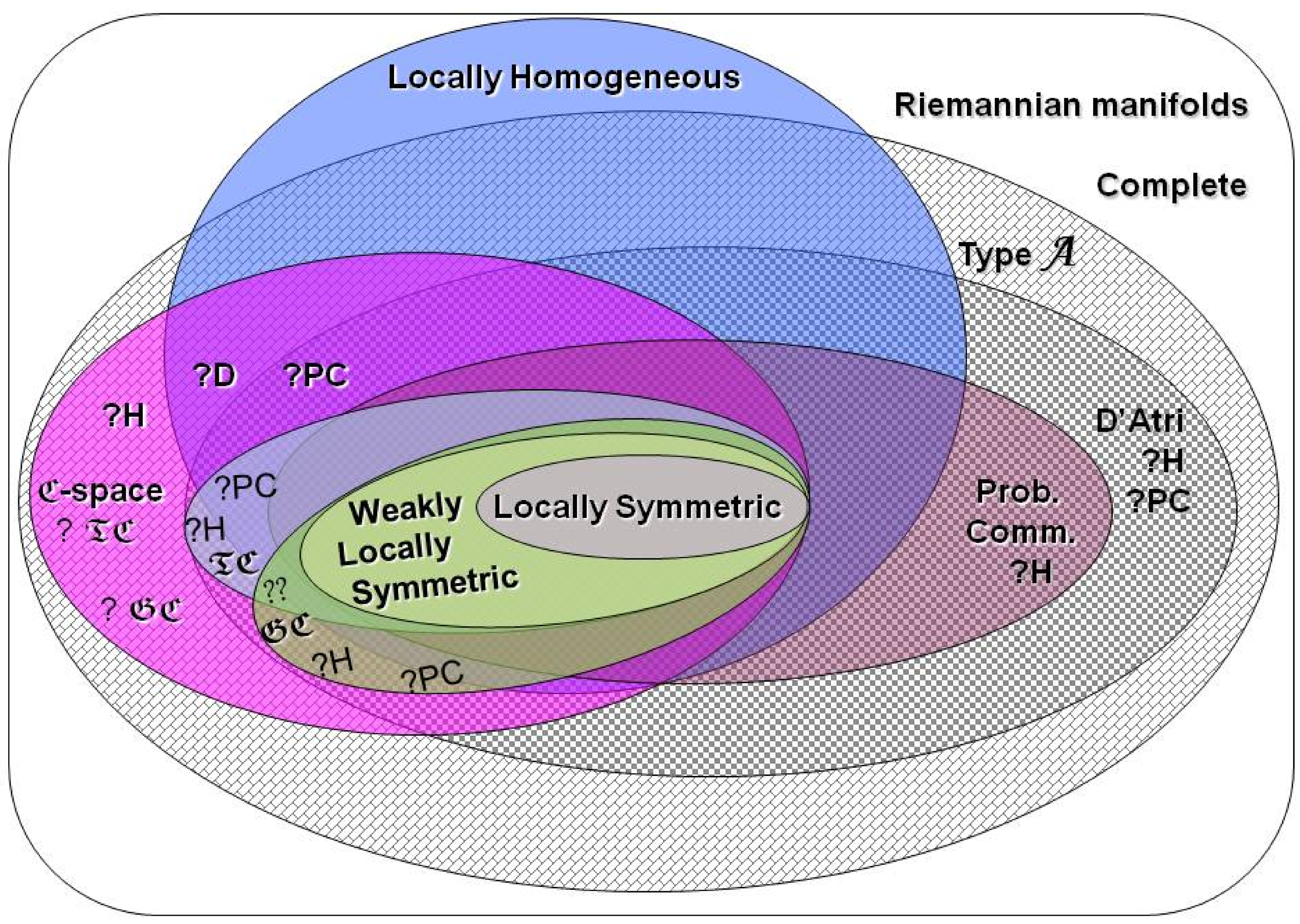
Abstract

Following Mark Kac, it is said that a geometric property of a compact Riemannian manifold can be heard if it can be determined from the eigenvalue spectrum of the associated Laplace operator on functions.

On the other hand, D'Atri spaces, manifolds of type $\mathcal{A}$, probabilistic commutative spaces, $\mathfrak{C}$-spaces, $\mathfrak{TC}$-spaces, and $\mathfrak{GC}$-spaces have been studied by many authors as symmetric-like Riemannian manifolds.

In this work, we prove that for closed Riemannian manifolds, none of the properties just mentioned can be heard. Another class of interest is the class of weakly symmetric manifolds. We consider the local version of this property and show that weak local symmetry is another inaudible property of Riemannian manifolds.

4. Relations between symmetric-like manifolds



5. Main Theorem

Each of the following properties is an inaudible property of Riemannian manifolds:

- Weak local symmetry,
- D'Atri property,
- probabilistic commutativity,
- the $\mathfrak{C}$ property,
- the $\mathfrak{TC}$ property,
- the $\mathfrak{GC}$ property,
- the type $\mathcal{A}$ property.

6. Sketch of the Proof

To prove the inaudibility of the previous curvature properties we use certain pairs of isospectral manifolds $N^{a+b,0}$ and $N^{a,b}$ for $a, b > 0$ introduced by Z.I. Szabó in [Sz] and we check that

- $N^{a+b,0}$ is weakly locally symmetric.
- $N^{a,b}$ is not even of type $\mathcal{A}$.

7. The family of closed Riemannian manifolds $N(j)$

Given $\mathfrak{v}$ and $\mathfrak{z}$ euclidean vector spaces, each endowed with a fixed inner product, and given $\mathcal{L}$ a cocompact lattice in $\mathfrak{z}$, one associates with any linear map $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ the following:

i) The two-step nilpotent metric Lie algebra $\mathfrak{g}(j)$

- With underlying vector space $\mathfrak{v} \oplus \mathfrak{z}$,
- whose inner product $\langle\ ,\ \rangle$ is given by letting $\mathfrak{v} \perp \mathfrak{z}$ and taking the given inner product on each factor,
- whose Lie bracket $[\ ,\ ]^j$ is defined by

$$[\mathfrak{v},\mathfrak{v}]^j \subseteq \mathfrak{z},\ \langle j_Z X, Y\rangle = \langle Z, [X,Y]^j\rangle$$

for all $X, Y \in \mathfrak{v}$ and $Z \in \mathfrak{z}$.

$\Rightarrow$

ii) The 2-step simply connected nilpotent Lie group $G(j)$

- Lie algebra is $\mathfrak{g}(j)$,
- The left invariant Riemannian metric $g(j)$ on $G(j)$ coincides with the chosen inner product on $\mathfrak{g}(j) = T_e G(j)$.

Remark 1 $\Rightarrow$

iii) The two-step nilpotent Lie group $G(j)/\exp^j(\mathcal{L})$

- $\exp^j(\mathcal{L})$ is a discrete central subgroup of $G(j)$

$\Rightarrow$ $g(j)$ induces a left invariant metric on $G(j)/\exp^j(\mathcal{L})$, which we denote again $g(j)$.

$\cup$

iv) $\tilde{N}(j) := \{\exp^j(X,Z) \mid X \in \mathfrak{v}, |X| = 1, Z \in \mathfrak{z}\}$

- $\tilde{N}(j)$ is a submanifold of $G(j)$.
- Endowed with the Riemannian metric induced by $g(j)$.

$\wr\wr$

$S^{\dim \mathfrak{v}-1} \times \mathfrak{z}$

Remark 2 $\Rightarrow$ Riemann covering

$\cup$

v) $N(j) := \tilde{N}(j)/\exp^j(\mathcal{L})$

- $N(j)$ is a submanifold of $G(j)/\exp^j(\mathcal{L})$.
- Endowed with the Riemannian metric induced by $g(j)$.
- $N(j)$ is compact.

$\wr\wr$

$S^{\dim \mathfrak{v}-1} \times (\mathfrak{z}/\mathcal{L})$

Remark 1.

- $\exp^j : \mathfrak{g}(j) \to G(j)$ is a diffeomorphism because $G(j)$ is simply connected and nilpotent.
- By Campbell-Baker-Hausdorff's formula

$$\exp^j(X,Z)\cdot\exp^j(Y,W) = \exp^j(X+Y, Z+W+\tfrac{1}{2}[X,Y]^j)$$

for all $X, Y \in \mathfrak{v}$ and $Z, W \in \mathfrak{z}$.

Remark 2.

- $\tilde{N}(j)$ is invariant under multiplication by elements of $\exp^j \mathfrak{z}$ because of

$$\exp^j(X,Z)\cdot\exp^j(0,W) = \exp^j(X, Z+W).$$

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