Local Audibility of a Hyperbolic Metric

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1. Introduction

(M^n, g) is a closed Riemannian manifold,

$$\Delta_g = -\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial}{\partial x^j}\right)$$

$$\mathrm{Sp}\,(\Delta_g) = \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \to +\infty\}$$

To which extent are the geometry and topology of a Riemannian manifold determined by the eigenvalue spectrum of its Laplacian?

M. Kac [1966]: Can one hear the shape of a drum?

Osgood–Philips–Sarnak [1988]: a Riemannian manifold is said to be audible, if it is determined by its spectrum uniquely up to an isometry.

2. Finiteness and compactness results. Finiteness conjecture

McKean [1974]: Within the class of two-dimensional Riemannian manifolds of constant negative Gaussian curvature, every isospectral family is finite if isometric surfaces are identified.

Osgood–Philips–Sarnak [1988]: Every isospectral set of metrics on a two-dimensional manifold is precompact in the C^∞-topology if isometric metrics are identified.

A similar compactness theorem for negatively curved 3-manifolds is proved by Brooks–Perry–Petersen [1992].

Conjecture 1 *Every isospectral family of metrics of negative Gaussian curvature on a compact orientable surface of genius ≥ 2 is finite if isometric metrics are identified.*

3. Local audibility

In virtue of the above-mentioned compactness theorem, the conjecture is equivalent to some local uniqueness statement. In connection with this, we introduce the following

Definition 2 *A Riemannian manifold (M, g) is said to be locally audible if there exists a neighborhood V of the metric g in the C^∞-topology such that every metric belonging to V and isospectral to g is isometric to g.*

Conjecture 1 is equivalent to the local audibility of a two-dimensional manifold of negative Gaussian curvature. To our opinion, the question on the local audibility of some Riemannian metric is of independent interest regardless to Conjecture 1. The main result of the present article is the following

Theorem 3 *A locally symmetric Riemannian manifold of negative sectional curvature is locally audible.*

4. Solenoidal tensor fields

While investigating the local audibility of a metric g, one has first of all to eliminate metrics that are isometric and close to g but do not coincide with g. Note that there are very many such metrics. Indeed, if a diffeomorphism $\varphi : M \to M$ is close to the identity then the metric $g' = \varphi^* g$ is isometric to g and close to it.

Given a Riemannian manifold (M, g), let $C^\infty(S^2\tau'_M)$ be the space of smooth symmetric rank two covariant tensor fields on M. The divergence $\delta_g : C^\infty(S^2\tau'_M) \to C^\infty(\tau'_M)$ is defined in coordinates by the equality $(\delta_g f)_i = g^{jk}\nabla_j f_{ik}$, where ∇ is the covariant derivative of the metric g. A tensor field f is said to be solenoidal if $\delta_g f = 0$. The above-mentioned elimination of "unnecessary" metrics is implemented with the help of the following

Lemma 4 *Croke–Dairbekov–Sh [2000]* *Let (M, g) be such that there exists at least one geodesic that is dense in the sphere bundle (this is true for a negatively curved manifold). For every $k \ge 2$ and $0 < \alpha < 1$, there exists a neighborhood $V \subset C^{k,\alpha}(S^2\tau'_M)$ of the metric g such that, for every metric $g' \in V$, there exists a diffeomorphism φ of the manifold M onto itself such that the tensor field $\varphi^* g'$ is solenoidal in the metric g, i.e., $\delta_g(\varphi^* g') = 0$. Moreover, the diffeomorphism φ can be chosen to be $C^{k,\alpha}$-close to the identity and φ is uniquely determined by the latter condition.*

In virtue of the lemma, Definition 2 takes the following equivalent form.

Proposition 5 *A negatively curved manifold* (M, g) *is locally audible if and only if the following statement is true. If* g_k $(k = 1, 2, \dots)$ *is a sequence of metrics on* M *converging to* g *in the* C^∞*-topology and such that every* g_k *is isospectral to* g *and satisfies* $\delta_g g_k = 0$ *then* $g_k = g$ *starting with some* k_0.

The latter statement is proved for negatively curved metrics if the sequence $g_k \to g$ is replaced with with a smooth one-parameter family g_t $(-\varepsilon < t < \varepsilon,\ g_0 = g)$, as is presented on the next slide.

5. Spectral rigidity

Definition 6 *A family g_t $(-\varepsilon < t < \varepsilon,\ g_0 = g)$ of metrics on a manifold M is called an isospectral deformation of the metric g if Sp(Δ_{g_t}) = Sp(Δ_g). The deformation is trivial if there exists a family of diffeomorphisms $\varphi_t : M \to M$ such that $g_t = \varphi_t^* g$. A Riemannian manifold is said to be spectrally rigid if it does not admit a nontrivial isospectral deformation.*

Theorem 7 *(Guillemin–Kazhdan [1980] for 2D-case, Croke-Sh [1998] in the general case)*
A negatively curved Riemannian manifold is spectrally rigid.

Theorem 8 *Croke–Sh [1998]: If a solenoidal tensor field $F \in C^\infty(S^2\tau'_M)$ on a negatively curved manifold integrates to zero over every closed geodesic, then $F \equiv 0$.*

6. A compactness estimate implies the local audibility

We use the following basic facts for negatively curved manifolds:

(1) Every free homotopic class contains a unique closed geodesic. The geodesic minimized the energy functional in its homotopic class.

(2) If eigenvalue spectra of two manifolds coincide, then their length spectra coincide too.

(3) If a solenoidal tensor field integrates to zero over every closed geodesic, then it is identical zero.

Let (M, g) be a Riemannian manifold of negative sectional curvature and g_m $(m = 1, 2, \dots)$ be a sequence of Riemannian metrics on M converging to g in the C^∞-topology. Assume every g_m to be isospectral to the metric g and solenoidal, i.e., $\delta_g g_m = 0$. In virtue of Proposition 5, we have to prove that g_m coincides with g starting with some m_0. We assume this false and try to get a contradiction. Passing to a subsequence, we can assume the tensor field $f_m = g_m - g$ to be not identically equal to zero for every n. Let γ be a closed geodesic of the metric g and γ_m be the closed geodesic of the metric g_m in the same free homotopy class as γ. Then γ_m converges uniformly to γ as $n \to \infty$. Since γ_m minimizes the energy functional E_{g_m} in its homotopy class, we can write

$$\oint\limits_\gamma f_m = \oint\limits_\gamma (g_m - g) = E_{g_m}(\gamma) - E_g(\gamma) \geq E_{g_m}(\gamma_m) - E_g(\gamma) = 0.$$

The last equality of the chain holds for a sufficiently large m since the metrics g_m and g have coincident length spectra.

Thus, for every closed geodesic γ of the metric g,

$$\oint\limits_{\gamma} f_m \geq 0 \quad \text{for} \quad m > m_0(\gamma) \tag{1}$$

Swopping the roles of g and g_m, we infer also that

$$\oint\limits_{\gamma_m} f_m \leq 0. \quad \text{for} \quad m > m_0(\gamma) \tag{2}$$

We normalize the tensor field f_m by setting $F_m = f_m/\|f_m\|_{H^k}$ with an appropriately chosen k. Inequalities (1.4)–(1.5) hold for F_m as well

$$\oint\limits_{\gamma} F_m \geq 0, \quad \oint\limits_{\gamma_m} F_m \leq 0 \quad \text{for } m \geq m_0(\gamma). \tag{3}$$

Assume for a moment the sequence F_m to converge in the H^k-norm: $\|F_m - F\|_{H^k} \to 0$ as $n \to \infty$. Passing to the limit in (3), we have

$$\oint\limits_{\gamma} F = 0 \tag{4}$$

for every closed geodesic γ of the metric g. Of course, F is a solenoidal tensor field. By Theorem 8, $F \equiv 0$. This contradicts to the equality $\|F\|_{H^k} = 1$.

The problem is thus reduced to the question: does the sequence F_m contain a subsequence converging in H^k? Since the embedding $H^{k+1} \subset H^k$ is compact, it suffices to prove the boundedness of the sequence F_m in the H^{k+1}-norm, $\|F_m\|_{H^{k+1}} \le C$. This means in terms of the sequence g_m that

$$\|g_m - g\|_{H^{k+1}} \le C\|g_m - g\|_{H^k}. \tag{5}$$

Compactness estimates like (5) appeared already in spectral geometry. Such an estimate (for $k = 0$) serves as a base for main results of Sh–Uhlmann [2000] and Dairbekov–Sh [2003] that are devoted to the spectral rigidity of Riemannian manifolds with the geodesic flow of Anosov type. Let $[g]$ be the set of all differences $g' - g$, where a metric g' is isospectral to g and satisfies $\delta_g g' = 0$. Roughly speaking, estimate (5) means that $[g] \cap V$ is a finite-dimensional set for a sufficiently small neighborhood of the origin $V \subset C^\infty(S^2\tau'_M)$. For example, if $[g] \cap V \subset W$ for some finite-dimensional space $W \subset C^\infty(S^2\tau'_M)$, then (5) holds since any two norms on W are equivalent.

7. Deriving a compactness estimate from heat invariants for a constant curvature metric

Let (M, g) be a Riemannian manifold and let $f \in C^\infty(S^2\tau'_M)$ be a sufficiently small solenoidal tensor field. Assume the metrics g and $g+f$ to be isospectral. Then, first of all, their volumes coincide. Equating the volumes, we obtain the estimate

$$\left|\int_M \mathrm{tr} f \, dV_0\right| \leq C\|f\|^2_{L^2} \tag{6}$$

Next, we equate heat invariants

$$a_{k+1}(M, g+f) - a_{k+1}(M, g) = 0. \tag{7}$$

We use the following representation of heat invariants Gilkey [1989]: for $k \geq 1$,

$$a_{k+1}(M,g) = \int_M \Big(c_k|\nabla^{(k-1)}S|^2 + c'_k|\nabla^{(k-1)}Ricc|^2 \\ - P_k(g,\nabla,R)\Big)\, dV,$$

where $P_k(g, \nabla, R)$ is some invariant polynomial in the variables $\nabla^{(l)}R$ $(l \leq k-2)$. It is a homogeneous polynomial of degree $2k+2$ in ∇ and R if the degree of homogeneity of ∇ is assumed to be equal to one and the degree of homogeneity of R, to two.

We expand the left-hand side of (7) into Tailor series in f and obtain with the help of Gilkey's representation

$$c_k\|f\|^2_{H^{k+1}} = \int_M P_k(R,f)\,dV, \qquad (8)$$

with some constant $c_k > 0$, where $P_k(R,f)$ is a power series in the curvature tensor R and tensor f and their covariant derivatives up to order k. We distinguish linear in f terms in $P_k(R,f)$

$$P_k(R,f) = L_k(R,f) + P'_k(R,f),$$

where $L_k(R,f)$ is a linear form in $\nabla^{(l)}f$ $(l \le k)$, and the series $P'_k(R,f)$ does not contain linear in f terms. For a sufficiently small f, the latter series admits the estimate

$$\int_M P'_k(R,f)\,dV \le C_k\|f\|^2_{H^k}$$

and we obtain from (8)

$$\|f\|^2_{H^{k+1}} \le C_k\|f\|^2_{H^k} + \int_M L_k(R,f)\,dV. \qquad (9)$$

The main difficulty of our approach relates to estimating the linear in f term $\int_M L_k(R,f)\,dV$ in (9). We can do this in the case of a constant curvature metric only. In this case, the linear form $L_k(R,f)=L_k(f)$ consists of summands that are obtained from derivatives

$$\nabla_{l_1\dots l_{2m}} f_{ij} \quad (2m \le k)$$

by raising a half of indices with the help of the tensor g^{ij} followed by the contraction in all indices grouped in pairs. For $m>0$, every such summand has obviously a divergent form and gives the zero contribution into integral (9). For $m=0$, we have the unique summand tr$f$ that is estimated by (6). Thus, (8) implies the compactness estimate

$$\|f\|^2_{H^{k+1}} \le C\|f\|^2_{H^k}. \qquad (8)$$

Lemma 9 *Let (M,g) be a Riemannian manifold of constant sectional curvature. For every $k\ge 2$, there exists a C^{k+1}-neighborhood V of the metric g such that the compactness estimate*

$$\|g'-g\|_{H^{k+1}} \le C\|g'-g\|_{H^k}$$

holds for every $g'\in V$ satisfying the conditions $\delta_g g'=0$, Vol$(M,g)=$ Vol(M,g') and $a_{k+1}(M,g)=a_{k+1}(M,g')$.

8. Why our approach does not work in the case of nonconstant negative curvature?

Given a Riemannian manifold (M, g) and tensor field $f \in C^{\infty}(S^2\tau'_M)$, the first variation of the spectral invariant a_k in the direction f is defined by

$$\dot{a}_k(M, g)f = da_k(M, g + tf)/dt|_{t=0}.$$

If the metric g has constant sectional curvature then

$$\dot{a}_k(M, g)f = 0 \tag{10}$$

for every f satisfying

$$2\dot{a}_0(M, g)f = \int\limits_M \mathrm{tr}\, f\, dV = 0. \tag{11}$$

This is a crucial fact for our approach. Indeed, in this case the difference

$$a_k(M, g + f) - a_k(M, g) \tag{12}$$

can be represented as a power series in f which does not contain linear terms. Leading terms of the series are quadratic in f that allows us to derive estimate The presence of linear in f terms in (12) stands as a stumbling block for our approach.

The author sees only one opportunity to fight with linear terms of series (3): the use of a linear combination of several invariants. Indeed, if a linear combination

$$a(M,g) = c_0 a_0(M,g) + \cdots + c_k a_k(M,g) \quad (c_k \neq 0) \tag{13}$$

with appropriately chosen constant coefficients turned out to have the zero first variation then we would be able to use the difference $a(M,g+f) - a(M,g)$ instead of (3). Unfortunately, a general metric g seems to have no linear combination like (13) satisfying $\dot a(M,g) = 0$. But if the curvature tensor of the metric g satisfies some natural differential equation, then such linear combinations probably may be found. In such a way, an opportunity arises for proving the local audibility of metrics belonging to some natural classes that are wider than the class of hyperbolic metrics.

The simplest of such equations is $\nabla R = 0$ that characterizes locally symmetric metrics. For such a metric, the first variations $\dot a_k(M,g)$ $(k = 1, 2\dots)$ live in a finite dimensional space and therefore they are linearly dependent. We thus obtain

Theorem 10 *A locally symmetric Riemannian manifold of negative sectional curvature is locally audible.*

In the above arguments, we have used only a finite subsystem of the system

$$\mathcal{F}_k(f) \equiv a_{k+1}(M, g+f) - a_{k+1}(M, g) = 0 \ (k = 1, 2, \dots). \tag{14}$$

Can the infinite system (14) be used for deriving a compactness estimate?

As we have seen, there is no problem if the gradients $\mathcal{F}'_k(0)$ $(k = 1, 2, \dots)$ are linearly dependent. So, let us assume the gradients to be linearly independent. In such a case, solutions of any finite subsystem of (14) constitute locally (in a neighborhood of the origin) a submanifold in $C^\infty(S^2\tau'_M)$. Is the same true for the infinite system (14)?

Problem 11 *Does the linear independence of gradients $\mathcal{F}'_k(0)$ $(k = 1, 2, \dots)$ imply that solutions to system (14) constitute locally (in a neighborhood of the origin) a smooth submanifold in the Frechét space $C^\infty(S^2\tau'_M)$?*

The set of solutions to system (14) is a locally compact subset of $C^\infty(S^2\tau'_M)$. This can be proved by the well known bootstrap argument. Thus, if the answer to Problem 11 is "yes", then the submanifold must be of a finite dimension. With the help of above arguments (choosing a convergent subsequence of $F_k = (g_k - g)/\|g_k - g\|$), this proves the local audibility of an arbitrary negatively curved manifold.

Probably some other spectral invariants, different of heat invariants, should be used for deriving a compactness estimate.

Presented results are published in [Sh, 2010].

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