Strichartz estimates for the Schrödinger equation on polygonal domains

Joint work with Matt Blair (UNM), G. Austin Ford (Northwestern U) and Sebastian Herr (U Bonn and U Düsseldorf) ... With a discussion of previous work with Andrew Hassell (Australian National University) and Luc Hillairet (Université Nantes)

Jeremy L. Marzuola

Mathematics Department
University of North Carolina, Chapel Hill

Dartmouth Spectral Geometry Workshop
July 23rd, 2010

Outline

The Set Up

- Let B be a planar, polygonal domain, not necessarily convex. Let V denote the set of vertices of B, and let Δ_B denote the Dirichlet or the Neumann Laplacian on $L^2(B)$.

The Nonconcentration Theorem

- **Theorem**

  Let B be as above and let U be any neighbourhood of V. Then there exists $c = c(U) > 0$ such that, for any L^2-normalized eigenfunction u of the Dirichlet (or Neumann) Laplacian Δ_B, we have

$$\int_U |u|^2 \geq c.$$

  That is, U is a control region for B.

Applicable Billiards

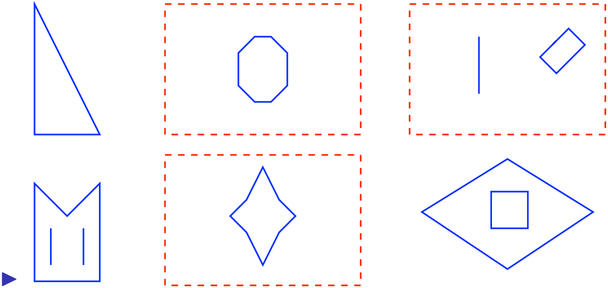


Figure: Examples of polygonal billiards for which the Theorem is applicable with Dirichlet or Neumann boudary conditions on the solid lines and periodic boundary conditions on the dashed lines.

Motivation

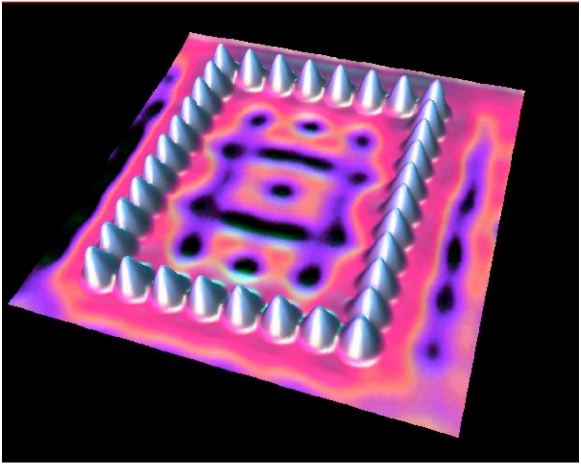
▶ 

Figure: A square billiard constructed by Crommie-Lutz-Eigler at IBM.

Motivation Cont.

- This goes back to the topic of Quantum Ergodicity, related to the question of Quantum/Classical Correspondence.
- Previous work in this area goes back to Burq-Zworski, Zelditch-Zworski, Gérard-Leichtman, Lindenstrauss, Sarnak, Melrose-Sjöstrand, de Verdiére,...
- Results proving “scarring” on hyperbolic manifolds like the quantum cat map have been studied in several results by Anantharaman and Nonnenmacher et al.

Motivation Cont.

- This goes back to the topic of Quantum Ergodicity, related to the question of Quantum/Classical Correspondence.
- Previous work in this area goes back to Burq-Zworski, Zelditch-Zworski, Gérard-Leichtman, Lindenstrauss, Sarnak, Melrose-Sjöstrand, de Verdiére,...
- Results proving "scarring" on hyperbolic manifolds like the quantum cat map have been studied in several results by Anantharaman and Nonnenmacher et al.

Motivation Cont.

- This goes back to the topic of Quantum Ergodicity, related to the question of Quantum/Classical Correspondence.
- Previous work in this area goes back to Burq-Zworski, Zelditch-Zworski, Gérard-Leichtman, Lindenstrauss, Sarnak, Melrose-Sjöstrand, de Verdiére,...
- Results proving “scarring” on hyperbolic manifolds like the quantum cat map have been studied in several results by Anantharaman and Nonnenmacher et al.

Partially Rectangular Billiards

- **Theorem (Burq-Zworski)**

  Let Ω be a partially rectangular billiard with the rectangular part $R \subset \Omega$, $\partial R = \Gamma_1 \cup \Gamma_2$, a decomposition into parallel components satisfying $\Gamma_2 \subset \partial\Omega$. Let Δ be the Dirichlet or Neumann Laplacian on Ω. Then for any *neighbourhood of Γ_1 in Ω, V, there exists C such that*

$$-\Delta u = \lambda u \implies \int_V |u(x)|^2 dx \geq \frac{1}{C} \int_R |u(x)|^2 dx \,,$$

  that is, no eigenfuction can concentrate in R and away from Γ_1.

Partially Rectangular Billiards Cont.

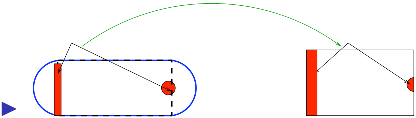


Figure: Control regions in which eigenfunctions have positive mass and the rectangular part for the Bunimovich stadium.

Motivation

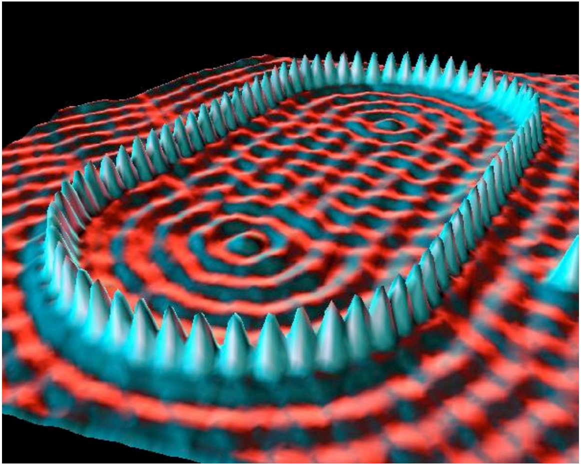


Figure: A stadium billiard constructed by a team at IBM.

Billiards with Obstacles

- Theorem (Burq-M-Zworski)

  Let V be any open neighbourhood of the convex boundary, $\partial\mathcal{O}$, in a Sinai billiard, S. If Δ is the Dirichlet or Neumann Laplace operator on S then there exists a constant, $C = C(V)$, such that

$$-h^2\Delta u = E(h)u \implies \int_V |u(x)|^2 dx \geq \frac{1}{C}\int_S |u(x)|^2 dx\,,$$

  for any h and $|E(h) - 1| < \frac{1}{2}$.

Nonconcentration Proof Ideas

- Assume there exists a sequence of eigenfunctions concentrating on a periodic orbit away from the obstacle.
- This trajectory can be trapped in a periodic cylinder.
- Contradiction argument using semiclassical defect measures and control theory estimates for solutions to inhomogeneous elliptic equations on rectangles.

Nonconcentration Proof Ideas

- Assume there exists a sequence of eigenfunctions concentrating on a periodic orbit away from the obstacle.
- This trajectory can be trapped in a periodic cylinder.
- Contradiction argument using semiclassical defect measures and control theory estimates for solutions to inhomogeneous elliptic equations on rectangles.

Nonconcentration Proof Ideas

- Assume there exists a sequence of eigenfunctions concentrating on a periodic orbit away from the obstacle.
- This trajectory can be trapped in a periodic cylinder.
- Contradiction argument using semiclassical defect measures and control theory estimates for solutions to inhomogeneous elliptic equations on rectangles.

Billiards with Obstacles

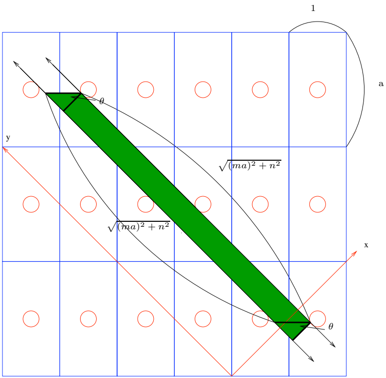


Figure: A maximal rectangle in a rational direction, avoiding the obstacle. Because the parrallelogram is certainly periodic and our region has uniform width, it is clear that the resulting rectangle is periodic.

Billiards with Slits

- Theorem (M)

  Let γ be an x-bounded trajectory on $P = \mathbb{T}^2 \setminus S$. If Δ is the Dirichlet Laplace operator on P then there exists no microlocal defect measure obtained from the eigenfunctions on P such that $\mathrm{supp}\,(d\mu) = \gamma$.

Billiards with Slits Cont.

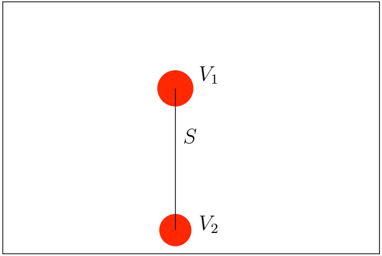
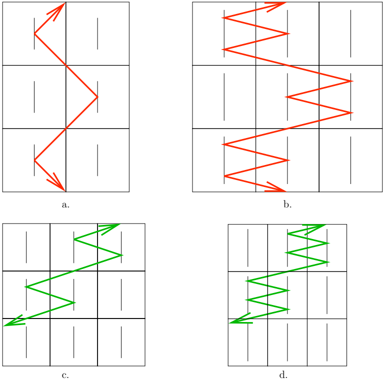


Figure: A pseudointegrable billiard P consisting of a torus with a slit, S, along which we have Dirichlet boundary conditions. We would like to show that eigenfunctions of the Laplacian on this torus must have concentration in the shaded regions V_1 and V_2.

Billiards with Slits Cont.



Above, a. and b. represent typical x-bounded trajectories, while c. and d. represent x-unbounded trajectories.

Figure: Some examples of x-bounded and x-unbounded trajectories.

Billiards with Slits Cont.

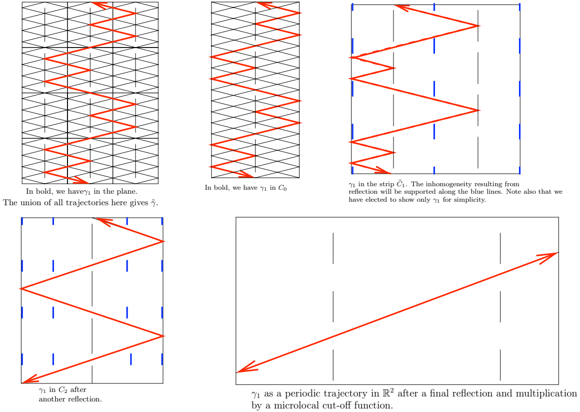


Figure: This diagram describes how we "unfold" the eigenfunctions in order to derive a contradiction.

Nonconcentration Proof Ideas

- Geometric condition classifying periodic orbits that miss the control region.
- Generalize to a Euclidean Surface with Conic Singularities with the geometric condition satisfied.
- Contradiction argument using semiclassical defect measures and control theory estimates for solutions to inhomogeneous elliptic equations on rectangles.

Nonconcentration Proof Ideas

- Geometric condition classifying periodic orbits that miss the control region.
- Generalize to a Euclidean Surface with Conic Singularities with the geometric condition satisfied.
- Contradiction argument using semiclassical defect measures and control theory estimates for solutions to inhomogeneous elliptic equations on rectangles.

Nonconcentration Proof Ideas

- Geometric condition classifying periodic orbits that miss the control region.
- Generalize to a Euclidean Surface with Conic Singularities with the geometric condition satisfied.
- Contradiction argument using semiclassical defect measures and control theory estimates for solutions to inhomogeneous elliptic equations on rectangles.

The Set Up for Strichartz

- Suppose $u(t,x) : [-T,T] \times \Omega \longrightarrow \mathbb{C}$ is a solution to the initial value problem for the Schrödinger equation on Ω:
- $$\begin{cases} (D_t + \Delta)\, u(t,x) = 0 \\ u(0,x) = f(x). \end{cases}$$
- Here, u satisfies either Dirichlet or Neumann homogeneous boundary conditions,

$$u\big|_{[-T,T]\times\partial\Omega} = 0 \qquad \text{or} \qquad \partial_n u\big|_{[-T,T]\times\partial\Omega} = 0.$$

The Set Up for Strichartz

- Suppose $u(t,x) : [-T,T] \times \Omega \longrightarrow \mathbb{C}$ is a solution to the initial value problem for the Schrödinger equation on Ω:
- $$\begin{cases} (D_t + \Delta)\, u(t,x) = 0 \\ u(0,x) = f(x). \end{cases}$$
- Here, u satisfies either Dirichlet or Neumann homogeneous boundary conditions,

$$u\big|_{[-T,T]\times\partial\Omega} = 0 \qquad \text{or} \qquad \partial_n u\big|_{[-T,T]\times\partial\Omega} = 0.$$

The Set Up for Strichartz

- Suppose $u(t,x) : [-T,T] \times \Omega \longrightarrow \mathbb{C}$ is a solution to the initial value problem for the Schrödinger equation on Ω:
- $$\begin{cases} (D_t + \Delta)\, u(t,x) = 0 \\ u(0,x) = f(x). \end{cases}$$
- Here, u satisfies either Dirichlet or Neumann homogeneous boundary conditions,

$$u\big|_{[-T,T]\times\partial\Omega} = 0 \qquad \text{or} \qquad \partial_n u\big|_{[-T,T]\times\partial\Omega} = 0.$$

The Strichartz Estimates

- These are a family of space-time integrability bounds of the form

$$\|u\|_{L^p([-T,T];L^q(\Omega))} \leq C_T \|f\|_{H^s(\Omega)}$$

with $p > 2$ and $\frac{2}{p} + \frac{2}{q} = 1$.

The Strichartz Estimates

- More precisely, this self-adjoint operator possesses a sequence of eigenfunctions forming a basis for $L^2(\Omega)$.
- We write the eigenfunction and eigenvalue pairs as $\Delta \varphi_j = \lambda_j^2 \varphi_j$, where λ_j denotes the frequency of vibration.
- The Sobolev space of order s can then be defined as the image of $L^2(\Omega)$ under $(1 + \Delta)^{-s}$ with norm

$$\|f\|_{H^s(\Omega)}^2 = \sum_{j=1}^{\infty} \left(1 + \lambda_j^2\right)^s |\langle f, \varphi_j \rangle|^2 .$$

- Here, $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product.

The Strichartz Estimates

- More precisely, this self-adjoint operator possesses a sequence of eigenfunctions forming a basis for $L^2(\Omega)$.
- We write the eigenfunction and eigenvalue pairs as $\Delta\varphi_j = \lambda_j^2 \varphi_j$, where λ_j denotes the frequency of vibration.
- The Sobolev space of order s can then be defined as the image of $L^2(\Omega)$ under $(1+\Delta)^{-s}$ with norm

$$\|f\|^2_{H^s(\Omega)} = \sum_{j=1}^{\infty} \left(1+\lambda_j^2\right)^s |\langle f, \varphi_j \rangle|^2 .$$

- Here, $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product.

The Strichartz Estimates

- More precisely, this self-adjoint operator possesses a sequence of eigenfunctions forming a basis for $L^2(\Omega)$.
- We write the eigenfunction and eigenvalue pairs as $\Delta \varphi_j = \lambda_j^2 \varphi_j$, where λ_j denotes the frequency of vibration.
- The Sobolev space of order s can then be defined as the image of $L^2(\Omega)$ under $(1 + \Delta)^{-s}$ with norm

$$\|f\|^2_{H^s(\Omega)} = \sum_{j=1}^{\infty} \left(1 + \lambda_j^2\right)^s |\langle f, \varphi_j \rangle|^2 .$$

- Here, $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product.

The Strichartz Estimates

- More precisely, this self-adjoint operator possesses a sequence of eigenfunctions forming a basis for $L^2(\Omega)$.
- We write the eigenfunction and eigenvalue pairs as $\Delta\varphi_j = \lambda_j^2\varphi_j$, where λ_j denotes the frequency of vibration.
- The Sobolev space of order s can then be defined as the image of $L^2(\Omega)$ under $(1+\Delta)^{-s}$ with norm

$$\|f\|_{H^s(\Omega)}^2 = \sum_{j=1}^{\infty} \left(1+\lambda_j^2\right)^s |\langle f, \varphi_j\rangle|^2 .$$

- Here, $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product.

The Strichartz Estimates

- Strichartz estimates are well-established when the domain Ω is replaced by Euclidean space.
- In that case, one can take $s = 0$, and by scaling considerations, this is the optimal order for the Sobolev space; see for example Strichartz (1977), Ginibre and Velo (1985), Keel and Tao (1998), etc.

The Strichartz Estimates

- Strichartz estimates are well-established when the domain Ω is replaced by Euclidean space.
- In that case, one can take $s = 0$, and by scaling considerations, this is the optimal order for the Sobolev space; see for example Strichartz (1977), Ginibre and Velo (1985), Keel and Tao (1998), etc.

The Strichartz Estimates

- When Ω is a compact domain or manifold, much less is known about the validity and optimality of these estimates.
- The finite volume of the manifold and the presence of trapped geodesics appear to limit the extent to which dispersion can occur.
- In addition, the imposition of boundary conditions complicate many of the known techniques for proving Strichartz estimates.
- Nonetheless, estimates on general compact domains with smooth boundary have been shown by Anton (2008) and Blair-Smith-Sogge (2008). Both of these works build on the approach for compact manifolds of Burq-Gérard-Tzvetkov (2004).

The Strichartz Estimates

- When Ω is a compact domain or manifold, much less is known about the validity and optimality of these estimates.
- The finite volume of the manifold and the presence of trapped geodesics appear to limit the extent to which dispersion can occur.
- In addition, the imposition of boundary conditions complicate many of the known techniques for proving Strichartz estimates.
- Nonetheless, estimates on general compact domains with smooth boundary have been shown by Anton (2008) and Blair-Smith-Sogge (2008). Both of these works build on the approach for compact manifolds of Burq-Gérard-Tzvetkov (2004).

The Strichartz Estimates

- When Ω is a compact domain or manifold, much less is known about the validity and optimality of these estimates.
- The finite volume of the manifold and the presence of trapped geodesics appear to limit the extent to which dispersion can occur.
- In addition, the imposition of boundary conditions complicate many of the known techniques for proving Strichartz estimates.
- Nonetheless, estimates on general compact domains with smooth boundary have been shown by Anton (2008) and Blair-Smith-Sogge (2008). Both of these works build on the approach for compact manifolds of Burq-Gérard-Tzvetkov (2004).

The Strichartz Estimates

- When Ω is a compact domain or manifold, much less is known about the validity and optimality of these estimates.
- The finite volume of the manifold and the presence of trapped geodesics appear to limit the extent to which dispersion can occur.
- In addition, the imposition of boundary conditions complicate many of the known techniques for proving Strichartz estimates.
- Nonetheless, estimates on general compact domains with smooth boundary have been shown by Anton (2008) and Blair-Smith-Sogge (2008). Both of these works build on the approach for compact manifolds of Burq-Gérard-Tzvetkov (2004).

The Theorem

- **Theorem**

  Let Ω be a compact polygonal domain in $\mathbb{R}^2$, and let Δ denote either the Dirichlet or Neumann Laplacian on Ω. Then for any solution $u = \exp(-it\Delta) f$ to the Schrödinger IBVP with f in $H^{\frac{1}{p}}(\Omega)$, the Strichartz estimates

$$\|u\|_{L^p([-T,T];L^q(\Omega))} \leq C_T \|f\|_{H^{\frac{1}{p}}(\Omega)}$$

  hold provided $p > 2$, $q \geq 2$, and $\frac{2}{p} + \frac{2}{q} = 1$.

Remarks

- In this work, the Neumann Laplacian is taken to be the Friedrichs extension of the Laplace operator acting on smooth functions which vanish in a neighborhood of the vertices.
- In this sense, our Neumann Laplacian imposes Dirichlet conditions at the vertices and Neumann conditions elsewhere.
- The Dirichlet Laplacian is taken to be the typical Friedrichs extension of the Laplace operator acting on smooth functions which are compactly supported in the interior of Ω.

Remarks

- In this work, the Neumann Laplacian is taken to be the Friedrichs extension of the Laplace operator acting on smooth functions which vanish in a neighborhood of the vertices.
- In this sense, our Neumann Laplacian imposes Dirichlet conditions at the vertices and Neumann conditions elsewhere.
- The Dirichlet Laplacian is taken to be the typical Friedrichs extension of the Laplace operator acting on smooth functions which are compactly supported in the interior of Ω.

Remarks

- In this work, the Neumann Laplacian is taken to be the Friedrichs extension of the Laplace operator acting on smooth functions which vanish in a neighborhood of the vertices.
- In this sense, our Neumann Laplacian imposes Dirichlet conditions at the vertices and Neumann conditions elsewhere.
- The Dirichlet Laplacian is taken to be the typical Friedrichs extension of the Laplace operator acting on smooth functions which are compactly supported in the interior of Ω.

Remarks

- We note that our estimates have a loss of $s = \frac{1}{p}$ derivatives as in Burq-Gérard-Tzvetkov (2004), which we believe is an artifact of our methods.
- Given specific geometries, there are results showing that such a loss is not sharp. For instance, when Ω is replaced by a flat rational torus, the Strichartz estimate with $p = q = 4$ holds for any $s > 0$, as was shown by Bourgain (1993); see also Bourgain (2007) for results in the case of irrational tori.
- However, we also point out that in certain geometries a loss of derivatives is expected due to the existence of gliding rays, as shown by Ivanovici (2008).

Remarks

- We note that our estimates have a loss of $s = \frac{1}{p}$ derivatives as in Burq-Gérard-Tzvetkov (2004), which we believe is an artifact of our methods.
- Given specific geometries, there are results showing that such a loss is not sharp. For instance, when Ω is replaced by a flat rational torus, the Strichartz estimate with $p = q = 4$ holds for any $s > 0$, as was shown by Bourgain (1993); see also Bourgain (2007) for results in the case of irrational tori.
- However, we also point out that in certain geometries a loss of derivatives is expected due to the existence of gliding rays, as shown by Ivanovici (2008).

Remarks

- We note that our estimates have a loss of $s = \frac{1}{p}$ derivatives as in Burq-Gérard-Tzvetkov (2004), which we believe is an artifact of our methods.
- Given specific geometries, there are results showing that such a loss is not sharp. For instance, when Ω is replaced by a flat rational torus, the Strichartz estimate with $p = q = 4$ holds for any $s > 0$, as was shown by Bourgain (1993); see also Bourgain (2007) for results in the case of irrational tori.
- However, we also point out that in certain geometries a loss of derivatives is expected due to the existence of gliding rays, as shown by Ivanovici (2008).

The Definition

- Definition

  A *Euclidean surface with conical singularities* (ESCS) is a topological space X possessing a decomposition $X = X_0 \sqcup P$ for a finite set of singular points $P \subsetneq X$ such that

  1. X_0 is an open, smooth two-dimensional Riemannian manifold with a locally Euclidean metric g, and
  2. each singular point p_j of P has a neighborhood U_j such that $U_j \setminus \{p_j\}$ is isometric to a neighborhood of the tip of a flat Euclidean cone $C(\mathbb{S}^1_{\rho_j})$ with p_j mapped to the cone tip.

The Real Theorem

- **Theorem**

  Let X be a compact ESCS, and let Δ_g be the Friedrichs extension of $\Delta_g\big|_{\mathcal{C}_c^\infty(X_0)}$. Then for any solution $u = \exp(-it\Delta_g)\, f$ to the Schrödinger IVP on X with initial data f in $H^{\frac{1}{p}}(X)$, the Strichartz estimates

  $$\|u\|_{L^p([-T,T];L^q(X))} \leq C_T \|f\|_{H^{\frac{1}{p}}(X)}$$

  hold provided $p > 2$, $q \geq 2$, and $\frac{2}{p} + \frac{2}{q} = 1$.

Littlewood-Paley

- Choose a nonnegative bump function β in $\mathcal{C}_c^\infty(\mathbb{R})$ supported in $\left(\frac{1}{4}, 4\right)$ and satisfying $\sum_{k\geq 1} \beta\left(2^{-k}\zeta\right) = 1$ for $\zeta \geq 1$.
- Taking $\beta_k(\zeta) \stackrel{\text{def}}{=} \beta(2^{-k}\zeta)$ for $k \geq 1$ and $\beta_0(\zeta) \stackrel{\text{def}}{=} 1 - \sum_{k\geq 1} \beta_k(\zeta)$, we define the frequency localization u_k of u in the spatial variable by

$$u_k \stackrel{\text{def}}{=} \beta_k\left(\sqrt{\Delta_g}\right) u.$$

- The operator $\beta_k(\sqrt{\Delta_g})$ is defined using the functional calculus with respect to Δ_g. Hence, $u = \sum_{k\geq 0} u_k$, and in particular, u_0 is localized to frequencies smaller than 1.

Littlewood-Paley

- Choose a nonnegative bump function β in $\mathcal{C}_c^\infty(\mathbb{R})$ supported in $\left(\frac{1}{4}, 4\right)$ and satisfying $\sum_{k \geq 1} \beta\left(2^{-k} \zeta\right) = 1$ for $\zeta \geq 1$.
- Taking $\beta_k(\zeta) \stackrel{\text{def}}{=} \beta\left(2^{-k} \zeta\right)$ for $k \geq 1$ and $\beta_0(\zeta) \stackrel{\text{def}}{=} 1 - \sum_{k \geq 1} \beta_k(\zeta)$, we define the frequency localization u_k of u in the spatial variable by

$$u_k \stackrel{\text{def}}{=} \beta_k\left(\sqrt{\Delta_g}\right) u.$$

- The operator $\beta_k(\sqrt{\Delta_g})$ is defined using the functional calculus with respect to Δ_g. Hence, $u = \sum_{k \geq 0} u_k$, and in particular, u_0 is localized to frequencies smaller than 1.

Littlewood-Paley

- Choose a nonnegative bump function β in $\mathcal{C}_c^\infty(\mathbb{R})$ supported in $\left(\frac{1}{4}, 4\right)$ and satisfying $\sum_{k\geq 1} \beta\left(2^{-k}\zeta\right) = 1$ for $\zeta \geq 1$.
- Taking $\beta_k(\zeta) \stackrel{\text{def}}{=} \beta\left(2^{-k}\zeta\right)$ for $k \geq 1$ and $\beta_0(\zeta) \stackrel{\text{def}}{=} 1 - \sum_{k\geq 1} \beta_k(\zeta)$, we define the frequency localization u_k of u in the spatial variable by

$$u_k \stackrel{\text{def}}{=} \beta_k\left(\sqrt{\Delta_{\mathrm{g}}}\right) u.$$

- The operator $\beta_k\left(\sqrt{\Delta_g}\right)$ is defined using the functional calculus with respect to Δ_g. Hence, $u = \sum_{k\geq 0} u_k$, and in particular, u_0 is localized to frequencies smaller than 1.

Square Function Estimates

- With this decomposition, we have the following square function estimate for elements a of $L^q(X)$,

$$\left\| \left(\sum_{k \geq 0} \left| \beta_k \left(\sqrt{\Delta_g} \right) a \right|^2 \right)^{\frac{1}{2}} \right\|_{L^q(X)} \approx \|a\|_{L^q(X)} ,$$

with implicit constants depending only on q.

Proof Using Square Function Estimates

- Delaying the proof of the square function estimate, we have by Minkowski's inequality that

$$\|u\|_{L^p([-T,T];L^q(X))} \lesssim \left(\sum_{k \geq 0} \|u_k\|^2_{L^p([-T,T];L^q(X))} \right)^{\frac{1}{2}}$$

since we are under the assumption that $p, q \geq 2$.

- We now claim that for each $k \geq 0$,

$$\|u_k\|_{L^p([-T,T];L^q(X))} \lesssim 2^{\frac{k}{p}} \|u_k(0,\cdot)\|_{L^2(X)} \, .$$

Proof Using Square Function Estimates

- Delaying the proof of the square function estimate, we have by Minkowski's inequality that

$$\|u\|_{L^p([-T,T];L^q(X))} \lesssim \left(\sum_{k \geq 0} \|u_k\|^2_{L^p([-T,T];L^q(X))} \right)^{\frac{1}{2}}$$

  since we are under the assumption that $p, q \geq 2$.
- We now claim that for each $k \geq 0$,

$$\|u_k\|_{L^p([-T,T];L^q(X))} \lesssim 2^{\frac{k}{p}} \|u_k(0,\cdot)\|_{L^2(X)} .$$

Proof Using Square Function Estimates

- Assuming this for the moment, we have by orthogonality and the localization of β that

$$2^{\frac{2k}{p}} \|u_k(0,\cdot)\|^2_{L^2(X)} = 2^{\frac{2k}{p}} \sum_{j=1}^{\infty} \beta_k(\lambda_j)^2 \left|\langle u(0,\cdot),\varphi_j\rangle\right|^2$$

$$\lesssim \sum_{j=1}^{\infty} \left(1+\lambda_j^2\right)^{1/p} \beta_k(\lambda_j)^2 \left|\langle u(0,\cdot),\varphi_j\rangle\right|^2 .$$

- We now sum this expression over k; after exchanging the order of summation in k and j, we obtain

$$\sum_{k\geq 0} 2^{\frac{2k}{p}} \|u_k(0,\cdot)\|^2_{L^2(X)} \lesssim \|u(0,\cdot)\|^2_{H^{\frac{1}{p}}(X)} .$$

- Combining this with Minkowski, we have reduced the proof of our Theorem to showing the dyadic Strichartz claim.

Proof Using Square Function Estimates

- Assuming this for the moment, we have by orthogonality and the localization of β that

$$2^{\frac{2k}{p}} \|u_k(0,\cdot)\|^2_{L^2(X)} = 2^{\frac{2k}{p}} \sum_{j=1}^{\infty} \beta_k(\lambda_j)^2 \left|\langle u(0,\cdot), \varphi_j\rangle\right|^2$$
$$\lesssim \sum_{j=1}^{\infty} \left(1+\lambda_j^2\right)^{1/p} \beta_k(\lambda_j)^2 \left|\langle u(0,\cdot), \varphi_j\rangle\right|^2 .$$

- We now sum this expression over k; after exchanging the order of summation in k and j, we obtain

$$\sum_{k\geq 0} 2^{\frac{2k}{p}} \|u_k(0,\cdot)\|^2_{L^2(X)} \lesssim \|u(0,\cdot)\|^2_{H^{\frac{1}{p}}(X)} .$$

- Combining this with Minkowski, we have reduced the proof of our Theorem to showing the dyadic Strichartz claim.

Proof Using Square Function Estimates

- Assuming this for the moment, we have by orthogonality and the localization of β that

$$2^{\frac{2k}{p}} \|u_k(0,\cdot)\|_{L^2(X)}^2 = 2^{\frac{2k}{p}} \sum_{j=1}^{\infty} \beta_k(\lambda_j)^2 \left|\langle u(0,\cdot), \varphi_j\rangle\right|^2$$

$$\lesssim \sum_{j=1}^{\infty} \left(1+\lambda_j^2\right)^{1/p} \beta_k(\lambda_j)^2 \left|\langle u(0,\cdot), \varphi_j\rangle\right|^2 .$$

- We now sum this expression over k; after exchanging the order of summation in k and j, we obtain

$$\sum_{k\geq 0} 2^{\frac{2k}{p}} \|u_k(0,\cdot)\|_{L^2(X)}^2 \lesssim \|u(0,\cdot)\|_{H^{\frac{1}{p}}(X)}^2 .$$

- Combining this with Minkowski, we have reduced the proof of our Theorem to showing the dyadic Strichartz claim.

Proof Using Square Function Estimates

- We observe that the claim follows from

$$\|u_k\|_{L^p([0,2^{-k}];L^q(X))} \lesssim \|u_k(0,\cdot)\|_{L^2(X)}.$$

- Indeed, if this estimate holds, then time translation and mass conservation imply the same estimate holds with the time interval $[0, 2^{-k}]$ replaced by $[2^{-k}m, 2^{-k}(m+1)]$. Taking a sum over all such dyadic intervals in $[-T, T]$ then yields the desired estimate.

Proof Using Square Function Estimates

- We observe that the claim follows from

$$\|u_k\|_{L^p([0,2^{-k}];L^q(X))} \lesssim \|u_k(0,\cdot)\|_{L^2(X)}.$$

- Indeed, if this estimate holds, then time translation and mass conservation imply the same estimate holds with the time interval $[0, 2^{-k}]$ replaced by $[2^{-k}m, 2^{-k}(m+1)]$. Taking a sum over all such dyadic intervals in $[-T, T]$ then yields the desired estimate.

Proof Using Square Function Estimates

- We localize our solution in space using a finite partition of unity $\sum_\ell \psi_\ell \equiv 1$ on X such that $\text{supp}(\psi_\ell)$ is contained in a neighborhood U_ℓ isometric to either an open subset of the plane $\mathbb{R}^2$ or a neighborhood of the tip of a Euclidean cone $C(\mathbb{S}^1_\rho)$.
- It now suffices to see that if ψ is an element of this partition and U denotes the corresponding open set in $\mathbb{R}^2$ or $C(\mathbb{S}^1_\rho)$, then

$$\|\psi\, u_k\|_{L^p([0,2^{-k}];L^q(U))} \lesssim \|u_k(0,\cdot)\|_{L^2(U)} .$$

Proof Using Square Function Estimates

- We localize our solution in space using a finite partition of unity $\sum_\ell \psi_\ell \equiv 1$ on X such that $\text{supp}(\psi_\ell)$ is contained in a neighborhood U_ℓ isometric to either an open subset of the plane $\mathbb{R}^2$ or a neighborhood of the tip of a Euclidean cone $C(\mathbb{S}^1_\rho)$.
- It now suffices to see that if ψ is an element of this partition and U denotes the corresponding open set in $\mathbb{R}^2$ or $C(\mathbb{S}^1_\rho)$, then

$$\|\psi\, u_k\|_{L^p([0,2^{-k}];L^q(U))} \lesssim \|u_k(0,\cdot)\|_{L^2(U)} .$$

Proof Using Square Function Estimates

- Observe that $\psi\, u_k$ solves the equation

$$\left(D_t + \Delta_{\mathrm{g}}\right)\left(\psi\, u_k\right) = \left[\Delta_{\mathrm{g}}, \psi\right] u_k$$

  over $\mathbb{R}^2$ or $C(\mathbb{S}^1_\rho)$.
- Letting $\mathbf{S}(t)$ denote the Schrödinger propagator either on Euclidean space or the Euclidean cone, depending on which space U lives in, we have for $t \geq 0$ that

$$\psi\, u_k(t,\cdot) = \mathbf{S}(t)\big(\psi\, u_k(0,\cdot)\big) + \int_0^{2^{-k}} \mathbf{1}_{\{t>s\}}(s)\,\mathbf{S}(t-s)\big(\left[\Delta_{\mathrm{g}}, \psi\right] u_k(s,\cdot)\big)\, \mathrm{d}s.$$

Proof Using Square Function Estimates

- Observe that $\psi\, u_k$ solves the equation

$$\left(D_t + \Delta_{\mathrm{g}}\right)\left(\psi\, u_k\right) = \left[\Delta_{\mathrm{g}}, \psi\right] u_k$$

over $\mathbb{R}^2$ or $C(\mathbb{S}^1_\rho)$.

- Letting $\mathbf{S}(t)$ denote the Schrödinger propagator either on Euclidean space or the Euclidean cone, depending on which space U lives in, we have for $t \geq 0$ that

$$\psi\, u_k(t,\cdot) = \mathbf{S}(t)\big(\psi\, u_k(0,\cdot)\big) + \int_0^{2^{-k}} \mathbf{1}_{\{t>s\}}(s)\, \mathbf{S}(t-s)\big(\left[\Delta_{\mathrm{g}}, \psi\right] u_k(s,\cdot)\big)\, \mathrm{d}s.$$

Proof Using Square Function Estimates

- On the plane, estimates on the Schrödinger operator are well known.
- On the cone, we apply the Strichartz estimates for the Schrödinger operator on the Euclidean cone without loss from Ford (2009).
- Once we have the estimates on the propagator, the dyadic Strichartz estimate follows in a standard fashion.

Proof Using Square Function Estimates

- On the plane, estimates on the Schrödinger operator are well known.
- On the cone, we apply the Strichartz estimates for the Schrödinger operator on the Euclidean cone without loss from Ford (2009).
- Once we have the estimates on the propagator, the dyadic Strichartz estimate follows in a standard fashion.

Proof Using Square Function Estimates

- On the plane, estimates on the Schrödinger operator are well known.
- On the cone, we apply the Strichartz estimates for the Schrödinger operator on the Euclidean cone without loss from Ford (2009).
- Once we have the estimates on the propagator, the dyadic Strichartz estimate follows in a standard fashion.

Proof of Square Function Estimates

- The estimate is actually valid for any exponent $1 < q < \infty$. If X_0 were compact, the estimate in Seeger-Sogge (1989) would suffice for our purpose.
- Extra care must be taken in our case, however, as X_0 is an incomplete manifold. Thus, we take advantage of a spectral multiplier theorem that allows us to employ a classical argument appearing in Stein (1970). This method is also treated in Ivanovici-Planchon (2008) and in the thesis of Blair (2005).

Proof of Square Function Estimates

- The estimate is actually valid for any exponent $1 < q < \infty$. If X_0 were compact, the estimate in Seeger-Sogge (1989) would suffice for our purpose.
- Extra care must be taken in our case, however, as X_0 is an incomplete manifold. Thus, we take advantage of a spectral multiplier theorem that allows us to employ a classical argument appearing in Stein (1970). This method is also treated in Ivanovici-Planchon (2008) and in the thesis of Blair (2005).

Proof of Square Function Estimates

- The multiplier theorem we use is due to Alexopolous (2004) and treats multipliers defined with respect to the spectrum of a differential operator on a manifold, see also the work of Duong, Ouhabaz, and Sikora (2002).
- It requires that the Riemannian measure is doubling and that the heat kernel $P(t,x,y)$ generated by Δ_g should satisfy a Gaussian upper bound of the form

$$P(t,x,y) \lesssim \frac{1}{\left|B\left(x,\sqrt{t}\right)\right|} \exp\left(-\frac{b \operatorname{dist}_g(x,y)^2}{t}\right),$$

where $\left|B\left(x,\sqrt{t}\right)\right|$ is the volume of the ball of radius $\sqrt{t}$ about x and $b > 0$ is a constant.
- We prove that this estimate holds on any ESCS.

Proof of Square Function Estimates

- The multiplier theorem we use is due to Alexopolous (2004) and treats multipliers defined with respect to the spectrum of a differential operator on a manifold, see also the work of Duong, Ouhabaz, and Sikora (2002).
- It requires that the Riemannian measure is doubling and that the heat kernel $P(t,x,y)$ generated by Δ_{g} should satisfy a Gaussian upper bound of the form

$$P(t,x,y) \lesssim \frac{1}{\left|B\left(x,\sqrt{t}\right)\right|} \exp\left(-\frac{b\,\mathrm{dist}_{\mathrm{g}}(x,y)^2}{t}\right),$$

where $\left|B\left(x,\sqrt{t}\right)\right|$ is the volume of the ball of radius $\sqrt{t}$ about x and $b > 0$ is a constant.
- We prove that this estimate holds on any ESCS.

Proof of Square Function Estimates

- The multiplier theorem we use is due to Alexopolous (2004) and treats multipliers defined with respect to the spectrum of a differential operator on a manifold, see also the work of Duong, Ouhabaz, and Sikora (2002).
- It requires that the Riemannian measure is doubling and that the heat kernel $P(t,x,y)$ generated by Δ_{g} should satisfy a Gaussian upper bound of the form

$$P(t,x,y) \lesssim \frac{1}{\left|B\left(x,\sqrt{t}\right)\right|} \exp\left(-\frac{b \operatorname{dist}_{\mathrm{g}}(x,y)^2}{t}\right),$$

  where $\left|B\left(x,\sqrt{t}\right)\right|$ is the volume of the ball of radius $\sqrt{t}$ about x and $b > 0$ is a constant.
- We prove that this estimate holds on any ESCS.

Proof of Square Function Estimates

- We use a theorem of Grigor'yan (1997) that establishes Gaussian upper bounds on arbitrary Riemannian manifolds.
- His result implies that if $P(t,x,y)$ satisfies on-diagonal bounds

$$P(t,x,x) \lesssim \max\left(\frac{1}{t}, C\right)$$

for some constant $C > 0$ then there exists $b > 0$ such that

$$P(t,x,y) \lesssim \max\left(\frac{1}{t}, C\right) \exp\left(-\frac{b \, \mathrm{dist}_g(x,y)^2}{t}\right).$$

- Since $\left|B\left(x,\sqrt{t}\right)\right| \approx t$ for bounded t, this is equivalent to the heat kernal bound.

Proof of Square Function Estimates

- We use a theorem of Grigor'yan (1997) that establishes Gaussian upper bounds on arbitrary Riemannian manifolds.
- His result implies that if $P(t,x,y)$ satisfies on-diagonal bounds

$$P(t,x,x) \lesssim \max\left(\frac{1}{t}, C\right)$$

for some constant $C > 0$ then there exists $b > 0$ such that

$$P(t,x,y) \lesssim \max\left(\frac{1}{t}, C\right) \exp\left(-\frac{b\,\mathrm{dist}_{\mathrm{g}}(x,y)^2}{t}\right).$$

- Since $\left|B\left(x,\sqrt{t}\right)\right| \approx t$ for bounded t, this is equivalent to the heat kernal bound.

Proof of Square Function Estimates

- We use a theorem of Grigor'yan (1997) that establishes Gaussian upper bounds on arbitrary Riemannian manifolds.
- His result implies that if $P(t,x,y)$ satisfies on-diagonal bounds

$$P(t,x,x) \lesssim \max\left(\frac{1}{t}, C\right)$$

for some constant $C > 0$ then there exists $b > 0$ such that

$$P(t,x,y) \lesssim \max\left(\frac{1}{t}, C\right) \exp\left(-\frac{b\, \mathrm{dist}_{\mathrm{g}}(x,y)^2}{t}\right).$$

- Since $\left|B\left(x,\sqrt{t}\right)\right| \approx t$ for bounded t, this is equivalent to the heat kernal bound.

Proof of Square Function Estimates

- The proof relies on an argument of Cheeger (1983) relating the heat kernel of a model space to that of an intrisic kernel on X_0.
- Then, on the Euclidean cone, we bound the heat kernel using an explicit formula for the heat kernel derived in Cheeger-Taylor I,II (1982) and specifically a form of the heat kernel writted down in Li (2003).

Proof of Square Function Estimates

- The proof relies on an argument of Cheeger (1983) relating the heat kernel of a model space to that of an intrisic kernel on X_0.
- Then, on the Euclidean cone, we bound the heat kernel using an explicit formula for the heat kernel derived in Cheeger-Taylor I,II (1982) and specifically a form of the heat kernel writted down in Li (2003).

Wave??

- The machinery presented here is generally applicable to any problem with know Strichartz estimates on the Euclidean cone.
- Hence, the authors hope to extend the result of Ford (2009) from the Schrödinger equation to the Wave equation, which our result then allows us to extend to any ESCS and hence any polygonal domain.

Wave??

- The machinery presented here is generally applicable to any problem with know Strichartz estimates on the Euclidean cone.
- Hence, the authors hope to extend the result of Ford (2009) from the Schrödinger equation to the Wave equation, which our result then allows us to extend to any ESCS and hence any polygonal domain.