Fourier transform, null variety, and Laplacian's eigenvalues

Michael Levitin

Reading University

Spectral Geometry Conference, 19 July 2010

joint work with Rafael Benguria (PUC Santiago) and Leonid Parnovski (UCL)

Objects of study in Benguria–L.–Parnovski (2009)

- $\Omega \subset \mathbb{R}^d$ — simply connected bounded domain with connected boundary $\partial\Omega$;

Objects of study in Benguria–L.–Parnovski (2009)

- $\Omega \subset \mathbb{R}^d$ — simply connected bounded domain with connected boundary $\partial\Omega$;
- $\chi_\Omega(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} \in \Omega, \\ 0 & \text{if } \mathbf{x} \notin \Omega \end{cases}$ — the characteristic function of Ω;

Objects of study in Benguria–L.–Parnovski (2009)

- $\Omega \subset \mathbb{R}^d$ — simply connected bounded domain with connected boundary $\partial\Omega$;
- $\chi_\Omega(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} \in \Omega, \\ 0 & \text{if } \mathbf{x} \notin \Omega \end{cases}$ — the characteristic function of Ω;
- $\widehat{\chi_\Omega}(\boldsymbol{\xi}) = \mathcal{F}[\chi_\Omega](\boldsymbol{\xi}) := \int_\Omega \mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{x}$ — its Fourier transform;

Objects of study in Benguria–L.–Parnovski (2009)

- $\Omega \subset \mathbb{R}^d$ — simply connected bounded domain with connected boundary $\partial\Omega$;
- $\chi_\Omega(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} \in \Omega, \\ 0 & \text{if } \mathbf{x} \notin \Omega \end{cases}$ — the characteristic function of Ω;
- $\widehat{\chi_\Omega}(\boldsymbol{\xi}) = \mathcal{F}[\chi_\Omega](\boldsymbol{\xi}) := \int_\Omega \mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{x}$ — its Fourier transform;
- $\mathcal{N}_\mathbb{C}(\Omega) := \{\boldsymbol{\xi} \in \mathbb{C}^d : \widehat{\chi_\Omega}(\boldsymbol{\xi}) = 0\}$ — its *complex null variety*, or *null set*;

Objects of study in Benguria–L.–Parnovski (2009)

- $\Omega \subset \mathbb{R}^d$ — simply connected bounded domain with connected boundary $\partial\Omega$;
- $\chi_\Omega(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} \in \Omega, \\ 0 & \text{if } \mathbf{x} \notin \Omega \end{cases}$ — the characteristic function of Ω;
- $\widehat{\chi_\Omega}(\boldsymbol{\xi}) = \mathcal{F}[\chi_\Omega](\boldsymbol{\xi}) := \int_\Omega \mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{x}$ — its Fourier transform;
- $\mathcal{N}_\mathbb{C}(\Omega) := \{\boldsymbol{\xi} \in \mathbb{C}^d : \widehat{\chi_\Omega}(\boldsymbol{\xi}) = 0\}$ — its *complex null variety*, or *null set*;
- $\kappa_\mathbb{C}(\Omega) := \operatorname{dist}(\mathcal{N}_\mathbb{C}(\Omega), \mathbf{0}) = \min\{|\boldsymbol{\xi}| : \boldsymbol{\xi} \in \mathcal{N}_\mathbb{C}(\Omega)\}$;

Objects of study in Benguria–L.–Parnovski (2009)

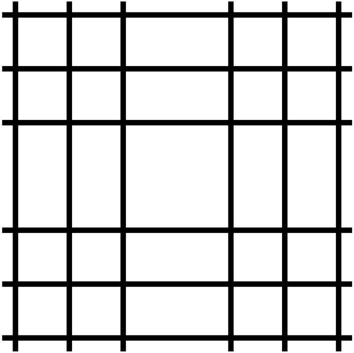
- $\Omega \subset \mathbb{R}^d$ — simply connected bounded domain with connected boundary $\partial\Omega$;
- $\chi_\Omega(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} \in \Omega, \\ 0 & \text{if } \mathbf{x} \notin \Omega \end{cases}$ — the characteristic function of Ω;
- $\widehat{\chi_\Omega}(\boldsymbol{\xi}) = \mathcal{F}[\chi_\Omega](\boldsymbol{\xi}) := \int_\Omega \mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{x}$ — its Fourier transform;
- $\mathcal{N}_\mathbb{C}(\Omega) := \{\boldsymbol{\xi} \in \mathbb{C}^d : \widehat{\chi_\Omega}(\boldsymbol{\xi}) = 0\}$ — its *complex null variety*, or *null set*;
- $\kappa_\mathbb{C}(\Omega) := \text{dist}(\mathcal{N}_\mathbb{C}(\Omega), \mathbf{0}) = \min\{|\boldsymbol{\xi}| : \boldsymbol{\xi} \in \mathcal{N}_\mathbb{C}(\Omega)\}$;
- Also, in particular for *balanced* (e.g. centrally symmetric domains) we look at $\mathcal{N}(\Omega) := \mathcal{N}_\mathbb{C}(\Omega) \cap \mathbb{R}^d = \{\boldsymbol{\xi} \in \mathbb{R}^d : \widehat{\chi_\Omega}(\boldsymbol{\xi}) = 0\} = \{\boldsymbol{\xi} \in \mathbb{R}^d : \int_\Omega \cos(\boldsymbol{\xi}\cdot\mathbf{x})\,\mathrm{d}\mathbf{x} = 0\}$ and $\kappa(\Omega) := \text{dist}(\mathcal{N}(\Omega), \mathbf{0})$;

Objects of study in Benguria–L.–Parnovski (2009)

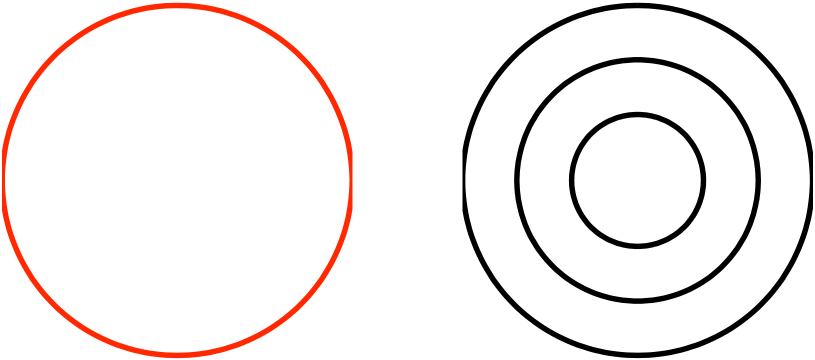
- $\Omega \subset \mathbb{R}^d$ — simply connected bounded domain with connected boundary $\partial\Omega$;
- $\chi_\Omega(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} \in \Omega, \\ 0 & \text{if } \mathbf{x} \notin \Omega \end{cases}$ — the characteristic function of Ω;
- $\widehat{\chi_\Omega}(\boldsymbol{\xi}) = \mathcal{F}[\chi_\Omega](\boldsymbol{\xi}) := \int_\Omega \mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{x}$ — its Fourier transform;
- $\mathcal{N}_\mathbb{C}(\Omega) := \{\boldsymbol{\xi} \in \mathbb{C}^d : \widehat{\chi_\Omega}(\boldsymbol{\xi}) = 0\}$ — its *complex null variety*, or *null set*;
- $\kappa_\mathbb{C}(\Omega) := \operatorname{dist}(\mathcal{N}_\mathbb{C}(\Omega), \mathbf{0}) = \min\{|\boldsymbol{\xi}| : \boldsymbol{\xi} \in \mathcal{N}_\mathbb{C}(\Omega)\}$;
- Also, in particular for *balanced* (e.g. centrally symmetric domains) we look at $\mathcal{N}(\Omega) := \mathcal{N}_\mathbb{C}(\Omega) \cap \mathbb{R}^d = \{\boldsymbol{\xi} \in \mathbb{R}^d : \widehat{\chi_\Omega}(\boldsymbol{\xi}) = 0\} = \{\boldsymbol{\xi} \in \mathbb{R}^d : \int_\Omega \cos(\boldsymbol{\xi}\cdot\mathbf{x})\,\mathrm{d}\mathbf{x} = 0\}$ and $\kappa(\Omega) := \operatorname{dist}(\mathcal{N}(\Omega), \mathbf{0})$;
- $(0 <)\lambda_1(\Omega) < \lambda_2(\Omega) \le \dots$ — Dirichlet Laplacian's eigenvalues, $(0 =)\mu_1(\Omega) < \mu_2(\Omega) \le \dots$ — Neumann Laplacian's eigenvalues.

What does $\mathcal{N}(\Omega)$ look like?

What does $\mathcal{N}(\Omega)$ look like?



What does $\mathcal{N}(\Omega)$ look like?



What does $\mathcal{N}(\Omega)$ look like?

Far field zero intensity diffraction pattern from the aperture Ω! Physical motivation.

What does $\mathcal{N}(\Omega)$ look like?

Far field zero intensity diffraction pattern from the aperture Ω! Physical motivation.

Also, it is of importance for *inverse problems* and image recognition. It is known that the structure of $\mathcal{N}(\Omega)$ *far from the origin* determines the shape of a *convex* set Ω.

Questions and Motivation

Main question: study $\kappa(\Omega)$ and its relations to the eigenvalues.

Questions and Motivation

Main question: study $\kappa(\Omega)$ and its relations to the eigenvalues.

Known links between $\mathcal{N}(\Omega)$ and spectral theory: two unsolved problems

Questions and Motivation

Main question: study $\kappa(\Omega)$ and its relations to the eigenvalues.
Known links between $\mathcal{N}(\Omega)$ and spectral theory: two unsolved problems

Pompeiu's Problem

Let $\mathcal{M}(d)$ be a group of rigid motions of $\mathbb{R}^d$, and Ω be a bounded simply connected domain with piecewise smooth connected boundary. Prove that the existence of a non-zero continuous function $f : \mathbb{R}^d \to \mathbb{R}^d$ such that $\int_{\mathbf{m}(\Omega)} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0$ for all $\mathbf{m} \in \mathcal{M}(d)$ implies that Ω is a ball.

Questions and Motivation

Main question: study $\kappa(\Omega)$ and its relations to the eigenvalues.
Known links between $\mathcal{N}(\Omega)$ and spectral theory: two unsolved problems

Pompeiu's Problem

Let $\mathcal{M}(d)$ be a group of rigid motions of $\mathbb{R}^d$, and Ω be a bounded simply connected domain with piecewise smooth connected boundary. Prove that the existence of a non-zero continuous function $f : \mathbb{R}^d \to \mathbb{R}^d$ such that $\int_{\mathbf{m}(\Omega)} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0$ for all $\mathbf{m} \in \mathcal{M}(d)$ implies that Ω is a ball.

Schiffer's conjecture

The existence of an eigenfunction v (corresponding to a non-zero eigenvalue μ) of a Neumann Laplacian on a domain Ω such that $v \equiv \mathrm{const}$ along the boundary $\partial\Omega$ (or, in other words, the existence of a non-constant solution v to the over-determined problem $-\Delta v = \mu v$, $\partial v/\partial n|_{\partial\Omega} = 0$, $v|_{\partial\Omega} = 1$) implies that Ω is a ball.

Motivation (contd.)

It is known that

Motivation (contd.)

It is known that

the positive answer to the Pompeiu problem

Motivation (contd.)

It is known that

the positive answer to the Pompeiu problem $\Longleftrightarrow$ Schiffer's conjecture

It is known that

the positive answer to the Pompeiu problem $\Longleftrightarrow$ Schiffer's conjecture $\Longleftrightarrow$ $\nexists$ Ω and $r > 0$ such that $\mathcal{N}_{\mathbb{C}}(\Omega) \supset \{\boldsymbol{\xi} \in \mathbb{C}^d : \sum_{j=1}^{d} \xi_j^2 = r^2\}$.

Motivation (contd.)

It is known that

the positive answer to the Pompeiu problem $\Longleftrightarrow$ Schiffer's conjecture $\Longleftrightarrow$ $\nexists$ Ω and $r > 0$ such that $\mathcal{N}_{\mathbb{C}}(\Omega) \supset \{\boldsymbol{\xi} \in \mathbb{C}^d : \sum_{j=1}^{d} \xi_j^2 = r^2\}$.

Thus, the interest in $\mathcal{N}_{\mathbb{C}}(\Omega)$. Also, it is of importance for *inverse problems* — determining the shape of Ω. A lot of publications, e.g. AGRANOVSKY, AVILES, BERENSTEIN, BROWN, KAHANE, SCHREIBER, TAYLOR, GAROFALO, SEGÀLA, T KOBAYASHI, ETC.

Motivation (contd.)

It is known that

the positive answer to the Pompeiu problem $\iff$ Schiffer's conjecture $\iff \nexists\ \Omega$ and $r > 0$ such that $\mathcal{N}_{\mathbb{C}}(\Omega) \supset \{\boldsymbol{\xi} \in \mathbb{C}^d : \sum_{j=1}^{d} \xi_j^2 = r^2\}$.

Thus, the interest in $\mathcal{N}_{\mathbb{C}}(\Omega)$. Also, it is of importance for *inverse problems* — determining the shape of Ω. A lot of publications, e.g. Agranovsky, Aviles, Berenstein, Brown, Kahane, Schreiber, Taylor, Garofalo, Segàla, T Kobayashi, etc.

Recall that we, on opposite, are interested only in the behaviour of $\mathcal{N}(\Omega)$ close to the origin, or more precisely in $\kappa(\Omega) = \operatorname{dist}(\mathcal{N}(\Omega), \mathbf{0})$

Motivation (contd.)

Theorem (Friedlander 1991)

For any $\Omega \subset \mathbb{R}^d$ with smooth boundary, any $k \geq 1$,

$$\mu_{k+1}(\Omega) < \lambda_k(\Omega)\,.$$

Motivation (contd.)

Theorem (Friedlander 1991)

For any $\Omega \subset \mathbb{R}^d$ *with smooth boundary, any* $k \geq 1$,

$$\mu_{k+1}(\Omega) < \lambda_k(\Omega)\,.$$

Theorem (Levine-Weinberger 1985)

If, additionally, Ω *is convex, then*

$$\mu_{k+d}(\Omega) < \lambda_k(\Omega)\,.$$

FILONOV's proof of FRIEDLANDER's Theorem

Proof.

Consider $\mathcal{L} = \{u_1, \ldots, u_k, \mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot\mathbf{x}}\}$, $|\boldsymbol{\xi}|^2 = \lambda_k$, $\boldsymbol{\xi} \in \mathbb{R}^d$, as a test space for μ_{k+1}, and calculate the Rayleigh ratios explicitly. All the non-sign-definite terms cancel out! □

FILONOV's proof of FRIEDLANDER's Theorem

Proof.

Consider $\mathcal{L} = \{u_1, \dots, u_k, \mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot\mathbf{x}}\}$, $|\boldsymbol{\xi}|^2 = \lambda_k$, $\boldsymbol{\xi} \in \mathbb{R}^d$, as a test space for μ_{k+1}, and calculate the Rayleigh ratios explicitly. All the non-sign-definite terms cancel out! □

In order to try to extend FILONOV's proof to establish $\mu_{k+d}(\Omega) < \lambda_k(\Omega)$, one may try to add extra exponentials to $\mathcal{L}$. Then, one needs inner products of exponentials to vanish - hence the need for estimates on $\kappa(\Omega)$.

Proof.

Consider $\mathcal{L} = \{u_1, \ldots, u_k, \mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot\mathbf{x}}\}$, $|\boldsymbol{\xi}|^2 = \lambda_k$, $\boldsymbol{\xi} \in \mathbb{R}^d$, as a test space for μ_{k+1}, and calculate the Rayleigh ratios explicitly. All the non-sign-definite terms cancel out! □

In order to try to extend FILONOV's proof to establish $\mu_{k+d}(\Omega) < \lambda_k(\Omega)$, one may try to add extra exponentials to $\mathcal{L}$. Then, one needs inner products of exponentials to vanish - hence the need for estimates on $\kappa(\Omega)$. In fact, if one knows that $\kappa(\Omega) \le 2\sqrt{\lambda_n(\Omega)}$, then one knows that $\mu_{k+2}(\Omega) \le \lambda_k(\Omega)$ holds for $k \ge n$.

First result

Lemma

For any $\Omega \subset \mathbb{R}^d$,

$$\kappa(\Omega) \geq \kappa_{\mathbb{C}}(\Omega) \geq \sqrt{\mu_2(\Omega)}\,.$$

First result

Lemma

For any $\Omega \subset \mathbb{R}^d$,

$$\kappa(\Omega) \geq \kappa_{\mathbb{C}}(\Omega) \geq \sqrt{\mu_2(\Omega)}\,.$$

Proof.

$\boldsymbol{\xi} \in \mathcal{N}_{\mathbb{C}}(\Omega)$, $v := \mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot\mathbf{x}}$

First result

Lemma

For any $\Omega \subset \mathbb{R}^d$,

$$\kappa(\Omega) \geq \kappa_{\mathbb{C}}(\Omega) \geq \sqrt{\mu_2(\Omega)}\,.$$

Proof.

$\boldsymbol{\xi} \in \mathcal{N}_{\mathbb{C}}(\Omega)$, $v := \mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot\mathbf{x}} \implies \langle v, 1\rangle_{L_2(\Omega)} = 0$ and $\|\mathbf{grad}\, v\|^2/\|v\|^2 = |\boldsymbol{\xi}|^2$. □

First result

Lemma

For any $\Omega \subset \mathbb{R}^d$,

$$\kappa(\Omega) \geq \kappa_{\mathbb{C}}(\Omega) \geq \sqrt{\mu_2(\Omega)}\,.$$

Proof.

$\boldsymbol{\xi} \in \mathcal{N}_{\mathbb{C}}(\Omega)$, $v := \mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot\mathbf{x}} \implies \langle v, 1\rangle_{L_2(\Omega)} = 0$ and $\|\mathbf{grad}\, v\|^2/\|v\|^2 = |\boldsymbol{\xi}|^2$. □

In fact, courtesy of FILONOV, we have

Lemma

For any $\Omega \subset \mathbb{R}^d$,

$$\kappa(\Omega) \geq 2\sqrt{\mu_2(\Omega)}\,.$$

Conjectures

From now on, we deal mostly with *convex* and *balanced* (e.g. centrally symmetric) domains.

Conjectures

From now on, we deal mostly with *convex* and *balanced* (e.g. centrally symmetric) domains. Notation: Ω^* is a ball of the same volume as Ω.

Conjectures

From now on, we deal mostly with *convex* and *balanced* (e.g. centrally symmetric) domains. Notation: Ω^* is a ball of the same volume as Ω.

Conjecture 1

If $\Omega \subset \mathbb{R}^d$ is convex and balanced, then

$$\kappa(\Omega) \leq \kappa(\Omega^*)\,, \tag{1}$$

with the equality iff Ω is a ball.

Conjectures

From now on, we deal mostly with *convex* and *balanced* (e.g. centrally symmetric) domains. Notation: Ω^* is a ball of the same volume as Ω.

Conjecture 1

If $\Omega \subset \mathbb{R}^d$ is convex and balanced, then

$$\kappa(\Omega) \le \kappa(\Omega^*)\,, \tag{1}$$

with the equality iff Ω is a ball.

Conjecture 2

If $\Omega \subset \mathbb{R}^d$ is convex and balanced, then

$$\kappa(\Omega) \le \sqrt{\lambda_2(\Omega)}\,, \tag{2}$$

with the equality iff Ω is a ball.

Why are the Conjectures plausible?

Balls

For a unit ball B_d, $\widehat{\chi_{B_d}}(\xi) = (2\pi)^{d/2} J_{d/2}(|\xi|)/|\xi|^{d/2}$,

Why are the Conjectures plausible?

Balls

For a unit ball B_d, $\widehat{\chi_{B_d}}(\xi) = (2\pi)^{d/2} J_{d/2}(|\xi|)/|\xi|^{d/2}$, and so

$$\kappa(B_d)^2 = j_{d/2,1}^2 = \lambda_2(B_d) = \lambda_3(B_d) = \cdots = \lambda_{1+d}(B_d)\,.$$

Why are the Conjectures plausible?

Balls

For a unit ball B_d, $\widehat{\chi_{B_d}}(\xi) = (2\pi)^{d/2} J_{d/2}(|\xi|)/|\xi|^{d/2}$, and so

$$\kappa(B_d)^2 = j^2_{d/2,1} = \lambda_2(B_d) = \lambda_3(B_d) = \cdots = \lambda_{1+d}(B_d)\,.$$

Boxes

For a parallelepiped P with sides $a_1 \geq a_2 \geq \cdots \geq a_d > 0$,

$$\lambda_2(P) = \pi^2 \left(4a_1^{-2} + (a_2)^{-2} + \cdots + (a_d)^{-2}\right) > 2\pi/a_1^2 = \kappa(P)^2\,.$$

Why are the Conjectures plausible?

Balls

For a unit ball B_d, $\widehat{\chi_{B_d}}(\xi) = (2\pi)^{d/2} J_{d/2}(|\xi|)/|\xi|^{d/2}$, and so

$$\kappa(B_d)^2 = j_{d/2,1}^2 = \lambda_2(B_d) = \lambda_3(B_d) = \cdots = \lambda_{1+d}(B_d)\,.$$

Boxes

For a parallelepiped P with sides $a_1 \geq a_2 \geq \cdots \geq a_d > 0$,

$$\lambda_2(P) = \pi^2 \left(4a_1^{-2} + (a_2)^{-2} + \cdots + (a_d)^{-2}\right) > 2\pi/a_1^2 = \kappa(P)^2\,.$$

Proving $\kappa(P) < \kappa(P^*)$ is already non-trivial.

Why are the Conjectures plausible?

Balls

For a unit ball B_d, $\widehat{\chi_{B_d}}(\boldsymbol{\xi}) = (2\pi)^{d/2} J_{d/2}(|\boldsymbol{\xi}|)/|\boldsymbol{\xi}|^{d/2}$, and so

$$\kappa(B_d)^2 = j_{d/2,1}^2 = \lambda_2(B_d) = \lambda_3(B_d) = \cdots = \lambda_{1+d}(B_d)\,.$$

Boxes

For a parallelepiped P with sides $a_1 \geq a_2 \geq \cdots \geq a_d > 0$,

$$\lambda_2(P) = \pi^2 \left(4a_1^{-2} + (a_2)^{-2} + \cdots + (a_d)^{-2}\right) > 2\pi/a_1^2 = \kappa(P)^2\,.$$

Proving $\kappa(P) < \kappa(P^*)$ is already non-trivial.

Numerics

Extensive numerical experiments. . .

Results

We cannot prove Conjectures as stated, and our results are only in $\mathbb{R}^2$.

Results

We cannot prove Conjectures as stated, and our results are only in $\mathbb{R}^2$.

Theorem

For any convex balanced $\Omega \subset \mathbb{R}^2$,

$$\kappa(\Omega) \leq C\kappa(\Omega^*)\,,$$

Results

We cannot prove Conjectures as stated, and our results are only in $\mathbb{R}^2$.

Theorem

For any convex balanced $\Omega \subset \mathbb{R}^2$,

$$\kappa(\Omega) \leq C\kappa(\Omega^*) , \qquad C = 2j_{0,1}/j_{1,1}$$

Results

We cannot prove Conjectures as stated, and our results are only in $\mathbb{R}^2$.

Theorem

For any convex balanced $\Omega \subset \mathbb{R}^2$,

$$\kappa(\Omega) \le C\kappa(\Omega^*)\,, \qquad C = 2j_{0,1}/j_{1,1} \approx 1.2552\,. \tag{3}$$

Results

We cannot prove Conjectures as stated, and our results are only in $\mathbb{R}^2$.

Theorem

For any convex balanced $\Omega \subset \mathbb{R}^2$,

$$\kappa(\Omega) \leq C\kappa(\Omega^*)\,, \qquad C = 2j_{0,1}/j_{1,1} \approx 1.2552\,. \tag{3}$$

Theorem

For any convex balanced $\Omega \subset \mathbb{R}^2$,

$$\kappa(\Omega) \leq 2\sqrt{\lambda_1(\Omega)}\,. \tag{4}$$

Results

We cannot prove Conjectures as stated, and our results are only in $\mathbb{R}^2$.

Theorem

For any convex balanced $\Omega \subset \mathbb{R}^2$,

$$\kappa(\Omega) \le C\kappa(\Omega^*)\,, \qquad C = 2j_{0,1}/j_{1,1} \approx 1.2552\,. \tag{3}$$

Theorem

For any convex balanced $\Omega \subset \mathbb{R}^2$,

$$\kappa(\Omega) \le 2\sqrt{\lambda_1(\Omega)}\,. \tag{4}$$

(4) follows from (3) and Faber-Krahn's $\lambda_1(\Omega) \le \lambda_1(\Omega^*)$.

Results (contd.)

Also, "near" the disk we can prove our original Conjectures for balanced star-shaped domains.

Results (contd.)

Also, "near" the disk we can prove our original Conjectures for balanced star-shaped domains. Let $F : S^1 \to \mathbb{R}$ be a C^2 function on the unit circle;

Results (contd.)

Also, "near" the disk we can prove our original Conjectures for balanced star-shaped domains. Let $F : S^1 \to \mathbb{R}$ be a C^2 function on the unit circle; $F(\theta + \pi) = F(\theta)$; $\int_0^{2\pi} F(\theta)\,\mathrm{d}\theta = 0$.

Results (contd.)

Also, "near" the disk we can prove our original Conjectures for balanced star-shaped domains. Let $F : S^1 \to \mathbb{R}$ be a C^2 function on the unit circle; $F(\theta + \pi) = F(\theta)$; $\int_0^{2\pi} F(\theta)\,\mathrm{d}\theta = 0$.

Define for $\epsilon \geq 0$, a domain in polar coordinates (r, θ) as

$$\Omega_{\epsilon F} := \{(r, \theta) \,:\, 0 \leq r \leq 1 + \epsilon F(\theta)\}\,.$$

By periodicity of F, $\Omega_{\epsilon F}$ is balanced, and also $\mathrm{vol}_2(\Omega_{\epsilon F}) = \pi + O(\epsilon^2)\,.$

Results (contd.)

Theorem

Let us fix a non-zero function F as above. Then

Theorem

Let us fix a non-zero function F *as above. Then*

$$\left.\frac{\mathrm{d}\kappa(\Omega_{\epsilon F})}{\mathrm{d}\epsilon}\right|_{\epsilon=+0} < 0\,,$$

Results (contd.)

Theorem

Let us fix a non-zero function F as above. Then

$$\left.\frac{\mathrm{d}\kappa(\Omega_{\epsilon F})}{\mathrm{d}\epsilon}\right|_{\epsilon=+0} < 0\,,$$

and

$$\left.\frac{\mathrm{d}\kappa(\Omega_{\epsilon F})}{\mathrm{d}\epsilon}\right|_{\epsilon=+0} < \left.\frac{\mathrm{d}\sqrt{\lambda_2(\Omega_{\epsilon F})}}{\mathrm{d}\epsilon}\right|_{\epsilon=+0}.$$

Results (contd.)

Theorem

Let us fix a non-zero function F as above. Then

$$\left.\frac{\mathrm{d}\kappa(\Omega_{\epsilon F})}{\mathrm{d}\epsilon}\right|_{\epsilon=+0} < 0\,,$$

and

$$\left.\frac{\mathrm{d}\kappa(\Omega_{\epsilon F})}{\mathrm{d}\epsilon}\right|_{\epsilon=+0} < \left.\frac{\mathrm{d}\sqrt{\lambda_2(\Omega_{\epsilon F})}}{\mathrm{d}\epsilon}\right|_{\epsilon=+0}\,.$$

Consequently, for sufficiently small $\epsilon > 0$ (depending on F), Conjectures 1 and 2 with $\Omega = \Omega_{\epsilon F}$ hold.

Results (contd.)

Theorem

Let us fix a non-zero function F *as above. Then*

$$\left.\frac{\mathrm{d}\kappa(\Omega_{\epsilon F})}{\mathrm{d}\epsilon}\right|_{\epsilon=+0} < 0\,,$$

and

$$\left.\frac{\mathrm{d}\kappa(\Omega_{\epsilon F})}{\mathrm{d}\epsilon}\right|_{\epsilon=+0} < \left.\frac{\mathrm{d}\sqrt{\lambda_2(\Omega_{\epsilon F})}}{\mathrm{d}\epsilon}\right|_{\epsilon=+0}\,.$$

Consequently, for sufficiently small $\epsilon > 0$ *(depending on* F*), Conjectures 1 and 2 with* $\Omega = \Omega_{\epsilon F}$ *hold.*

The same is true for $\epsilon < 0$.

Results (contd.)

On the other hand, fixing ϵ and varying F produces a non-convex counter-example:

Results (contd.)

On the other hand, fixing ϵ and varying F produces a non-convex counter-example:

Theorem

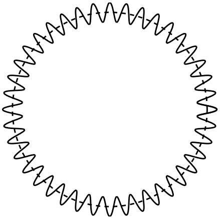
For each positive $\tilde{\delta}$ there exists a star-shaped balanced domain Ω with $\mathrm{vol}_2(\Omega) = \pi$ and such that $B(0, 1 - \tilde{\delta}) \subset \Omega \subset B(0, 1 + \tilde{\delta})$, for which $\kappa(\Omega) > j_{1,1}$.

Results (contd.)

On the other hand, fixing ϵ and varying F produces a non-convex counter-example:

Theorem

For each positive $\tilde{\delta}$ there exists a star-shaped balanced domain Ω with $\text{vol}_2(\Omega) = \pi$ and such that $B(0, 1 - \tilde{\delta}) \subset \Omega \subset B(0, 1 + \tilde{\delta})$, for which $\kappa(\Omega) > j_{1,1}$.



Results (contd.)

We can also prove the original Conjectures for sufficiently elongated convex balanced planar domains.

Theorem (also by ZASTAVNYI, 1984)

Suppose that $d = 2$ and $D(\Omega)$ is the diameter of Ω. Then

$$\kappa(\Omega) \leq \frac{4\pi}{D(\Omega)}\,.$$

Results (contd.)

We can also prove the original Conjectures for sufficiently elongated convex balanced planar domains.

Theorem (also by Zastavnyi, 1984)

Suppose that $d = 2$ and $D(\Omega)$ is the diameter of Ω. Then

$$\kappa(\Omega) \leq \frac{4\pi}{D(\Omega)}\,.$$

Corollary

Conjecture 1 holds for convex, balanced domains $\Omega \subset \mathbb{R}^2$ such that

$$\frac{\sqrt{\pi}\,D(\Omega)}{2\sqrt{\mathrm{vol}_2(\Omega)}} \geq \frac{2\pi}{j_{1,1}}$$

Results (contd.)

We can also prove the original Conjectures for sufficiently elongated convex balanced planar domains.

Theorem (also by ZASTAVNYI, 1984)

Suppose that $d = 2$ and $D(\Omega)$ is the diameter of Ω. Then

$$\kappa(\Omega) \le \frac{4\pi}{D(\Omega)}\,.$$

Corollary

Conjecture 1 holds for convex, balanced domains $\Omega \subset \mathbb{R}^2$ such that

$$\frac{\sqrt{\pi}D(\Omega)}{2\sqrt{\mathrm{vol}_2(\Omega)}} \ge \frac{2\pi}{j_{1,1}} \approx 1.6398\,.$$

Ideas of the proofs I

Fix the *direction* $\mathbf{e} \in S^{d-1}$ of the Fourier variable $\boldsymbol{\xi} = \rho\mathbf{e}$, and look at the ρ-roots of

$$\widehat{\chi}_{\mathbf{e}}(\rho) := \widehat{\chi}(\rho\mathbf{e}) = \int_\Omega \cos(\rho\mathbf{e}\cdot\mathbf{x})\,\mathrm{d}\mathbf{x}\,.$$

Ideas of the proofs I

Fix the *direction* $\mathbf{e} \in S^{d-1}$ of the Fourier variable $\boldsymbol{\xi} = \rho\mathbf{e}$, and look at the ρ-roots of

$$\widehat{\chi}_{\mathbf{e}}(\rho) := \widehat{\chi}(\rho\mathbf{e}) = \int_\Omega \cos(\rho\mathbf{e}\cdot\mathbf{x})\,\mathrm{d}\mathbf{x}\,.$$

Let $\kappa_j(\mathbf{e})$ be the j-th ρ-root of $\widehat{\chi}_{\mathbf{e}}(\rho)$. Then $\kappa(\Omega) = \min_{\mathbf{e}\in S^{d-1}} \kappa_1(\mathbf{e})$.

Ideas of the proofs I

Fix the *direction* $\mathbf{e} \in S^{d-1}$ of the Fourier variable $\boldsymbol{\xi} = \rho\mathbf{e}$, and look at the ρ-roots of

$$\widehat{\chi}_{\mathbf{e}}(\rho) := \widehat{\chi}(\rho\mathbf{e}) = \int_\Omega \cos(\rho\mathbf{e}\cdot\mathbf{x})\,\mathrm{d}\mathbf{x}\,.$$

Let $\kappa_j(\mathbf{e})$ be the j-th ρ-root of $\widehat{\chi}_{\mathbf{e}}(\rho)$. Then $\kappa(\Omega) = \min_{\mathbf{e}\in S^{d-1}} \kappa_1(\mathbf{e})$.

Lemma

Let $d = 2$, then

$$\kappa_j(\mathbf{e}) \le \frac{\pi(j+1)}{w(\mathbf{e})}\,.$$

where $w(\mathbf{e})$ is a half-breadth of Ω in direction $\mathbf{e}$.

Ideas of the proofs I

Fix the *direction* $\mathbf{e} \in S^{d-1}$ of the Fourier variable $\boldsymbol{\xi} = \rho\mathbf{e}$, and look at the ρ-roots of

$$\widehat{\chi}_{\mathbf{e}}(\rho) := \widehat{\chi}(\rho\mathbf{e}) = \int_{\Omega} \cos(\rho\mathbf{e}\cdot\mathbf{x})\,\mathrm{d}\mathbf{x}\,.$$

Let $\kappa_j(\mathbf{e})$ be the j-th ρ-root of $\widehat{\chi}_{\mathbf{e}}(\rho)$. Then $\kappa(\Omega) = \min\limits_{\mathbf{e}\in S^{d-1}} \kappa_1(\mathbf{e})$.

Lemma

Let $d = 2$, then

$$\kappa_j(\mathbf{e}) \le \frac{\pi(j+1)}{w(\mathbf{e})}\,.$$

where $w(\mathbf{e})$ is a half-breadth of Ω in direction $\mathbf{e}$.

Not optimal!

Ideas of the proofs I

Fix the *direction* $\mathbf{e} \in S^{d-1}$ of the Fourier variable $\boldsymbol{\xi} = \rho\mathbf{e}$, and look at the ρ-roots of

$$\widehat{\chi}_{\mathbf{e}}(\rho) := \widehat{\chi}(\rho\mathbf{e}) = \int_\Omega \cos(\rho\mathbf{e}\cdot\mathbf{x})\,\mathrm{d}\mathbf{x}\,.$$

Let $\kappa_j(\mathbf{e})$ be the j-th ρ-root of $\widehat{\chi}_{\mathbf{e}}(\rho)$. Then $\kappa(\Omega) = \min_{\mathbf{e}\in S^{d-1}} \kappa_1(\mathbf{e})$.

Lemma

Let $d = 2$, then

$$\kappa_j(\mathbf{e}) \le \frac{\pi(j+1)}{w(\mathbf{e})}\,.$$

where $w(\mathbf{e})$ is a half-breadth of Ω in direction $\mathbf{e}$.

Not optimal! Not true if $d \ge 3$!

Ideas of the proofs I

Fix the *direction* $\mathbf{e} \in S^{d-1}$ of the Fourier variable $\boldsymbol{\xi} = \rho\mathbf{e}$, and look at the ρ-roots of

$$\widehat{\chi}_{\mathbf{e}}(\rho) := \widehat{\chi}(\rho\mathbf{e}) = \int_\Omega \cos(\rho\mathbf{e}\cdot\mathbf{x})\,\mathrm{d}\mathbf{x}\,.$$

Let $\kappa_j(\mathbf{e})$ be the j-th ρ-root of $\widehat{\chi}_{\mathbf{e}}(\rho)$. Then $\kappa(\Omega) = \min\limits_{\mathbf{e}\in S^{d-1}} \kappa_1(\mathbf{e})$.

Lemma

Let $d = 2$, then

$$\kappa_j(\mathbf{e}) \le \frac{\pi(j+1)}{w(\mathbf{e})}\,.$$

where $w(\mathbf{e})$ is a half-breadth of Ω in direction $\mathbf{e}$.

Not optimal! Not true if $d \ge 3$! Still, gives the above Theorems for planar "cigars".

Ideas of the proofs II

We want to find $\mathbf{e} \in S^1$ and $\tau > 0$ such that $\widehat{\chi}_{\mathbf{e}}(\tau) < 0$; then we know $\tau > \kappa(\Omega)$.

We want to find $\mathbf{e} \in S^1$ and $\tau > 0$ such that $\widehat{\chi}_{\mathbf{e}}(\tau) < 0$; then we know $\tau > \kappa(\Omega)$.

Let us instead seek τ such that

$$0 > \int_{S^1} \int_{\Omega} \cos(\tau \mathbf{e} \cdot \mathbf{x}) \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{e}$$

Ideas of the proofs II

We want to find $\mathbf{e} \in S^1$ and $\tau > 0$ such that $\widehat{\chi}_{\mathbf{e}}(\tau) < 0$; then we know $\tau > \kappa(\Omega)$.

Let us instead seek τ such that

$$0 > \int_{S^1} \int_{\Omega} \cos(\tau \mathbf{e} \cdot \mathbf{x}) \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{e} = \int_{\Omega} J_0(\tau |\mathbf{x}|) \, \mathrm{d}\mathbf{x} \, .$$

Ideas of the proofs II

We want to find $\mathbf{e} \in S^1$ and $\tau > 0$ such that $\widehat{\chi}_{\mathbf{e}}(\tau) < 0$; then we know $\tau > \kappa(\Omega)$.

Let us instead seek τ such that

$$0 > \int_{S^1} \int_\Omega \cos(\tau \mathbf{e} \cdot \mathbf{x})\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{e} = \int_\Omega J_0(\tau|\mathbf{x}|)\, \mathrm{d}\mathbf{x}\,.$$

We characterize convex balanced Ω by either

$$\eta(r;\Omega) := \mathrm{vol}_1(\Omega \cap \{|\mathbf{x}| = r\})$$

or

$$\alpha(r;\Omega) := \frac{1}{\pi}\int_0^r \eta(\rho;\Omega)\,\mathrm{d}\rho = \frac{1}{\pi}\mathrm{vol}_2(\Omega \cap B_2(r))$$

and numbers

$$r_- = r_-(\Omega) = \min_{\mathbf{e}\in S^1} w(\mathbf{e})\,, \qquad r_+ := \max_{\mathbf{e}\in S^1} w(\mathbf{e})\,.$$

Obviously, r_- is the inradius of Ω and $2r_+$ is its diameter.

Ideas of the proofs II (contd.)

Some properties of the functions η and α and the numbers $r_\pm$ are obvious:

Ideas of the proofs II (contd.)

Some properties of the functions η and α and the numbers $r_\pm$ are obvious:

- $\eta(r)$ and $\alpha(r)$ are non-negative; also $\alpha(r)$ is non-decreasing;

Ideas of the proofs II (contd.)

Some properties of the functions η and α and the numbers $r_\pm$ are obvious:

- $\eta(r)$ and $\alpha(r)$ are non-negative; also $\alpha(r)$ is non-decreasing;
- $\eta(r) \equiv 2\pi r$ and $\alpha(r) \equiv r^2$ for $r \leq r_-$;

Ideas of the proofs II (contd.)

Some properties of the functions η and α and the numbers $r_\pm$ are obvious:

- $\eta(r)$ and $\alpha(r)$ are non-negative; also $\alpha(r)$ is non-decreasing;
- $\eta(r) \equiv 2\pi r$ and $\alpha(r) \equiv r^2$ for $r \leq r_-$;
- $\eta(r) \equiv 0$ and $\alpha(r) \equiv \text{const} = \text{vol}_2(\Omega)/\pi$ for $r \geq r_+$

Ideas of the proofs II (contd.)

Some properties of the functions η and α and the numbers $r_\pm$ are obvious:

- $\eta(r)$ and $\alpha(r)$ are non-negative; also $\alpha(r)$ is non-decreasing;
- $\eta(r) \equiv 2\pi r$ and $\alpha(r) \equiv r^2$ for $r \le r_-$;
- $\eta(r) \equiv 0$ and $\alpha(r) \equiv \mathrm{const} = \mathrm{vol}_2(\Omega)/\pi$ for $r \ge r_+$

An additional important property is valid for *planar* convex domains.

Lemma

Let $\Omega \subset \mathbb{R}^2$ be a balanced convex domain. Then for $r \in [r_-(\Omega), r_+(\Omega)]$, the function $\eta(r)$ is decreasing and the function $\alpha(r)$ is concave.

Ideas of the proofs II (contd.)

Some properties of the functions η and α and the numbers $r_\pm$ are obvious:

- $\eta(r)$ and $\alpha(r)$ are non-negative; also $\alpha(r)$ is non-decreasing;
- $\eta(r) \equiv 2\pi r$ and $\alpha(r) \equiv r^2$ for $r \leq r_-$;
- $\eta(r) \equiv 0$ and $\alpha(r) \equiv \text{const} = \text{vol}_2(\Omega)/\pi$ for $r \geq r_+$

An additional important property is valid for *planar* convex domains.

Lemma

Let $\Omega \subset \mathbb{R}^2$ be a balanced convex domain. Then for $r \in [r_-(\Omega), r_+(\Omega)]$, the function $\eta(r)$ is decreasing and the function $\alpha(r)$ is concave.

Question

Is it true for $d \geq 3$?

Ideas of the proofs II (contd.)

Some properties of the functions η and α and the numbers $r_\pm$ are obvious:

- $\eta(r)$ and $\alpha(r)$ are non-negative; also $\alpha(r)$ is non-decreasing;
- $\eta(r) \equiv 2\pi r$ and $\alpha(r) \equiv r^2$ for $r \le r_-$;
- $\eta(r) \equiv 0$ and $\alpha(r) \equiv \text{const} = \text{vol}_2(\Omega)/\pi$ for $r \ge r_+$

An additional important property is valid for *planar* convex domains.

Lemma

Let $\Omega \subset \mathbb{R}^2$ be a balanced convex domain. Then for $r \in [r_-(\Omega), r_+(\Omega)]$, the function $\eta(r)$ is decreasing and the function $\alpha(r)$ is concave.

Question

Is it true for $d \ge 3$? No! Extensive study of $\eta(r)$ and generalizations in a recent paper by Campi, Gardenr, Gronchi

After some change of variables and integration by parts, our Theorem 1 reduces to

Ideas of the proofs II (contd.)

After some change of variables and integration by parts, our Theorem 1 reduces to

Problem

For $I[\alpha] := \int_0^{j_{0,3}} \alpha(r) J_1(r) \mathrm{d}r$, show that

$$\sup_{\alpha \in \mathcal{A}} I[\alpha] < 0\,,$$

where the class $\mathcal{A}$ consists of continuous functions $\alpha : [0, j_{0,3}] \to \mathbb{R}$ satisfying

Ideas of the proofs II (contd.)

After some change of variables and integration by parts, our Theorem 1 reduces to

Problem

For $I[\alpha] := \int_0^{j_{0,3}} \alpha(r) J_1(r) \mathrm{d}r$, show that

$$\sup_{\alpha \in \mathcal{A}} I[\alpha] < 0 \,,$$

where the class $\mathcal{A}$ consists of continuous functions $\alpha : [0, j_{0,3}] \to \mathbb{R}$ satisfying

(a) $\alpha(r)$ is non-negative and non-decreasing;

(b) $\alpha(r) = r^2/(4j_{0,1}^2)$ for $0 \le r \le r_-$;

(c) $\alpha(r) = 1$ for $r \ge r_+$;

(d) $\alpha(r)$ is concave for $r_- \le r \le r_+$;

(e) $j_{0,1}^2/2 < r_- \le 2j_{0,1} \le r_+ < 2\pi$.

Ideas of the proofs II (contd.)

Proof is technical, difficult, does not extend to dimensions higher than two, and eventually reduces to showing that

Ideas of the proofs II (contd.)

Proof is technical, difficult, does not extend to dimensions higher than two, and eventually reduces to showing that

Ideas of the proofs II (contd.)

Proof is technical, difficult, does not extend to dimensions higher than two, and eventually reduces to showing that $Ly_- + M$ is negative, where

$$L := J_0\left(\frac{\tau^2}{8}\right) - \frac{1}{2\pi - j_{1,1}}\left(\pi^2 J_1(2\pi)\mathbf{H}_0(2\pi) - \pi^2 J_0(2\pi)\mathbf{H}_1(2\pi)\right.$$
$$\left. + \frac{\pi j_{1,1}}{2} J_0(j_{1,1})\mathbf{H}_1(j_{1,1}) + j_{1,1} J_0(j_{1,1}) + 2\pi J_0(2\pi)\right)$$

$$M := \frac{1}{8} J_2\left(\frac{\tau^2}{8}\right) + \frac{1}{2\pi - j_{1,1}}\left(\pi^2 J_1(2\pi)\mathbf{H}_0(2\pi) - \pi^2 J_0(2\pi)\mathbf{H}_1(2\pi)\right.$$
$$\left. + \frac{\pi j_{1,1}}{2} J_0(j_{1,1})\mathbf{H}_1(j_{1,1}) - j_{1,1} J_0(j_{1,1}) + 2\pi J_0(2\pi)\right);$$

$$\tau := 2 j_{0,1}; \qquad y_- := 1 - \frac{(2\pi - j_{1,1})(64 - \tau^2)}{8(16\pi - \tau^2)}.$$

Ideas of the proofs II (contd.)

Proof is technical, difficult, does not extend to dimensions higher than two, and eventually reduces to showing that $-0.00724446126 = Ly_- + M$ is negative, where

$$L := J_0\left(\frac{\tau^2}{8}\right) - \frac{1}{2\pi - j_{1,1}}\left(\pi^2 J_1(2\pi)\mathbf{H}_0(2\pi) - \pi^2 J_0(2\pi)\mathbf{H}_1(2\pi) + \frac{\pi j_{1,1}}{2} J_0(j_{1,1})\mathbf{H}_1(j_{1,1}) + j_{1,1}J_0(j_{1,1}) + 2\pi J_0(2\pi)\right)$$

$$M := \frac{1}{8}J_2\left(\frac{\tau^2}{8}\right) + \frac{1}{2\pi - j_{1,1}}\left(\pi^2 J_1(2\pi)\mathbf{H}_0(2\pi) - \pi^2 J_0(2\pi)\mathbf{H}_1(2\pi) + \frac{\pi j_{1,1}}{2} J_0(j_{1,1})\mathbf{H}_1(j_{1,1}) - j_{1,1}J_0(j_{1,1}) + 2\pi J_0(2\pi)\right);$$

$$\tau := 2j_{0,1}; \qquad y_- := 1 - \frac{(2\pi - j_{1,1})(64 - \tau^2)}{8(16\pi - \tau^2)}.$$

Dropping restrictions

Dropping central symmetry

Dropping restrictions

Dropping central symmetry

Let $T = T_{1,a}$ be a right-angled triangle with sides 1, $a > 1$, and $\sqrt{1 + a^2}$.
Then $\kappa(T_{1,a}) = 2\pi\sqrt{1 + a^{-2}}$.

Dropping restrictions

Dropping central symmetry

Let $T = T_{1,a}$ be a right-angled triangle with sides 1, $a > 1$, and $\sqrt{1+a^2}$. Then $\kappa(T_{1,a}) = 2\pi\sqrt{1+a^{-2}}$. Both Conjectures with $\Omega = T$ hold for values of a sufficiently close to one, but fail for large a or small a.

Dropping restrictions

Dropping central symmetry

Let $T = T_{1,a}$ be a right-angled triangle with sides 1, $a > 1$, and $\sqrt{1+a^2}$. Then $\kappa(T_{1,a}) = 2\pi\sqrt{1+a^{-2}}$. Both Conjectures with $\Omega = T$ hold for values of a sufficiently close to one, but fail for large a or small a. May still hold for $\kappa_{\mathbb{C}}(T)$!

Dropping restrictions

Dropping central symmetry

Let $T = T_{1,a}$ be a right-angled triangle with sides 1, $a > 1$, and $\sqrt{1 + a^2}$. Then $\kappa(T_{1,a}) = 2\pi\sqrt{1 + a^{-2}}$. Both Conjectures with $\Omega = T$ hold for values of a sufficiently close to one, but fail for large a or small a. May still hold for $\kappa_{\mathbb{C}}(T)$!

Dropping convexity

Dropping restrictions

Dropping central symmetry

Let $T = T_{1,a}$ be a right-angled triangle with sides 1, $a > 1$, and $\sqrt{1 + a^2}$. Then $\kappa(T_{1,a}) = 2\pi\sqrt{1 + a^{-2}}$. Both Conjectures with $\Omega = T$ hold for values of a sufficiently close to one, but fail for large a or small a. May still hold for $\kappa_{\mathbb{C}}(T)$!

Dropping convexity

There is no C such that $\kappa(\Omega) \leq C\kappa(\Omega^*)$ holds uniformly for all balanced connected two-dimensional domains Ω.

Dropping restrictions

Dropping central symmetry

Let $T = T_{1,a}$ be a right-angled triangle with sides 1, $a > 1$, and $\sqrt{1+a^2}$. Then $\kappa(T_{1,a}) = 2\pi\sqrt{1+a^{-2}}$. Both Conjectures with $\Omega = T$ hold for values of a sufficiently close to one, but fail for large a or small a. May still hold for $\kappa_{\mathbb{C}}(T)$!

Dropping convexity

There is no C such that $\kappa(\Omega) \leq C\kappa(\Omega^*)$ holds uniformly for all balanced connected two-dimensional domains Ω.

Conclusions

- New object: $\kappa(\Omega)$;

Conclusions

- New object: $\kappa(\Omega)$;
- any dimension d and any Ω, $2\sqrt{\mu_2(\Omega)} \leq \kappa(\Omega)$;

Conclusions

- New object: $\kappa(\Omega)$;
- any dimension d and any Ω, $2\sqrt{\mu_2(\Omega)} \leq \kappa(\Omega)$;
- $d = 2$ and convex, balanced Ω, $\kappa(\Omega) < 2\sqrt{\lambda_1(\Omega)}$;

Conclusions

- New object: $\kappa(\Omega)$;
- any dimension d and any Ω, $2\sqrt{\mu_2(\Omega)} \leq \kappa(\Omega)$;
- $d = 2$ and convex, balanced Ω, $\kappa(\Omega) < 2\sqrt{\lambda_1(\Omega)}$;
- conjecture that for any d, and convex, balanced Ω, $\kappa(\Omega) < \sqrt{\lambda_2(\Omega)}$;

Conclusions

- New object: $\kappa(\Omega)$;
- any dimension d and any Ω, $2\sqrt{\mu_2(\Omega)} \leq \kappa(\Omega)$;
- $d = 2$ and convex, balanced Ω, $\kappa(\Omega) < 2\sqrt{\lambda_1(\Omega)}$;
- conjecture that for any d, and convex, balanced Ω, $\kappa(\Omega) < \sqrt{\lambda_2(\Omega)}$;
- Many open problems!