Quasi-normal modes for rotating black holes

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Quasi-normal modes (QNM's) are the frequencies of gravitational waves emitted by a black hole. They have been the object of research of many physicists, but there has been only a handful of attempts to put this research on a mathematical foundation. The presented work focuses on the Kerr-de Sitter model of a slowly rotating black hole; the goals are to:

- **Obtain a rigorous definition of QNM's**
- **Establish connections of QNM's to expansions of linear waves**
- **Study the asymptotic distribution of QNM's and compare it to the numerical results in the physics literature**

The first goal, together with partial results on the other two, such as exponential decay of local energy for the wave equation, is achieved in [D]. (See Theorems 1 and 2 on the left board.) The third goal is work in progress, with some results presented below.

Separation of variables

The Kerr–de Sitter metric is invariant under rotation about a certain axis; denote by $\mathcal{D}'_k$ the space of functions with angular momentum $k \in \mathbb{Z}$. If $P_g(\omega)$ is the stationary d'Alembert–Beltrami operator, then

$$P_g(\omega)|_{\mathcal{D}'_k} = P_r(\omega, k) + P_\theta(\omega),$$

where $P_r(\omega, k)$ is a differential operator in the radial variable only and $P_\theta(\omega)$ is an operator on the sphere $\mathbb{S}^2$. This is the Teukolsky separation of variables.

Let ω be a QNM. Then the Laurent decomposition of $R_g(\omega)$ provides a certain solution u to the equation $P_g(\omega)u = 0$; using the separation of variables, we can write $u = u_r \otimes u_\theta$, where

$$(P_r(\omega, k) + \lambda)u_r = 0, \qquad (3)$$

$$(P_\theta(\omega, k) - \lambda)u_\theta = 0, \ u_\theta \in \mathcal{D}'_k \qquad (4)$$

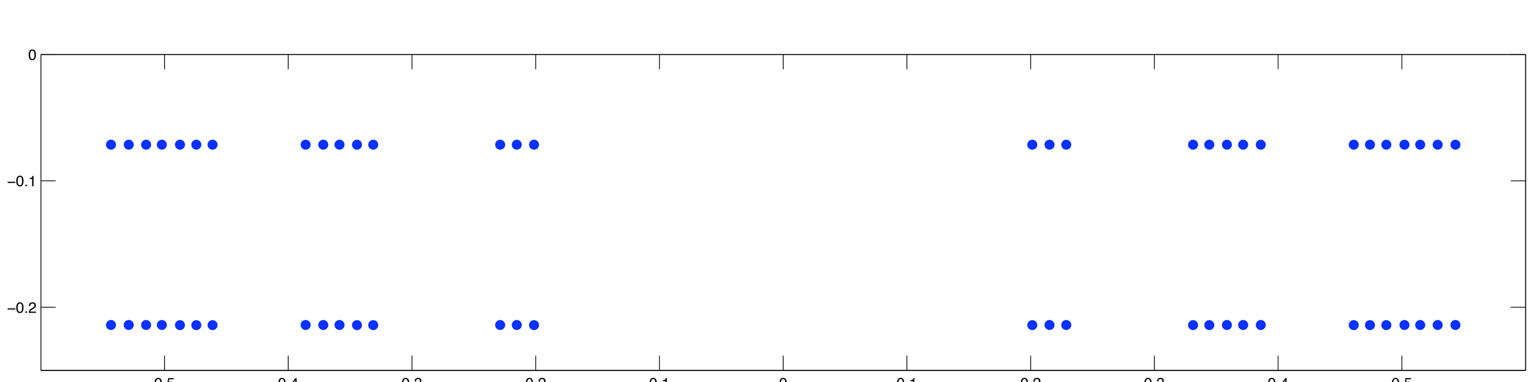
for some $k \in \mathbb{Z}, \lambda \in \mathbb{C}$. So, ω is a QNM iff we can solve problems (3) and (4) simultaneously for some k and λ.

Spectral results

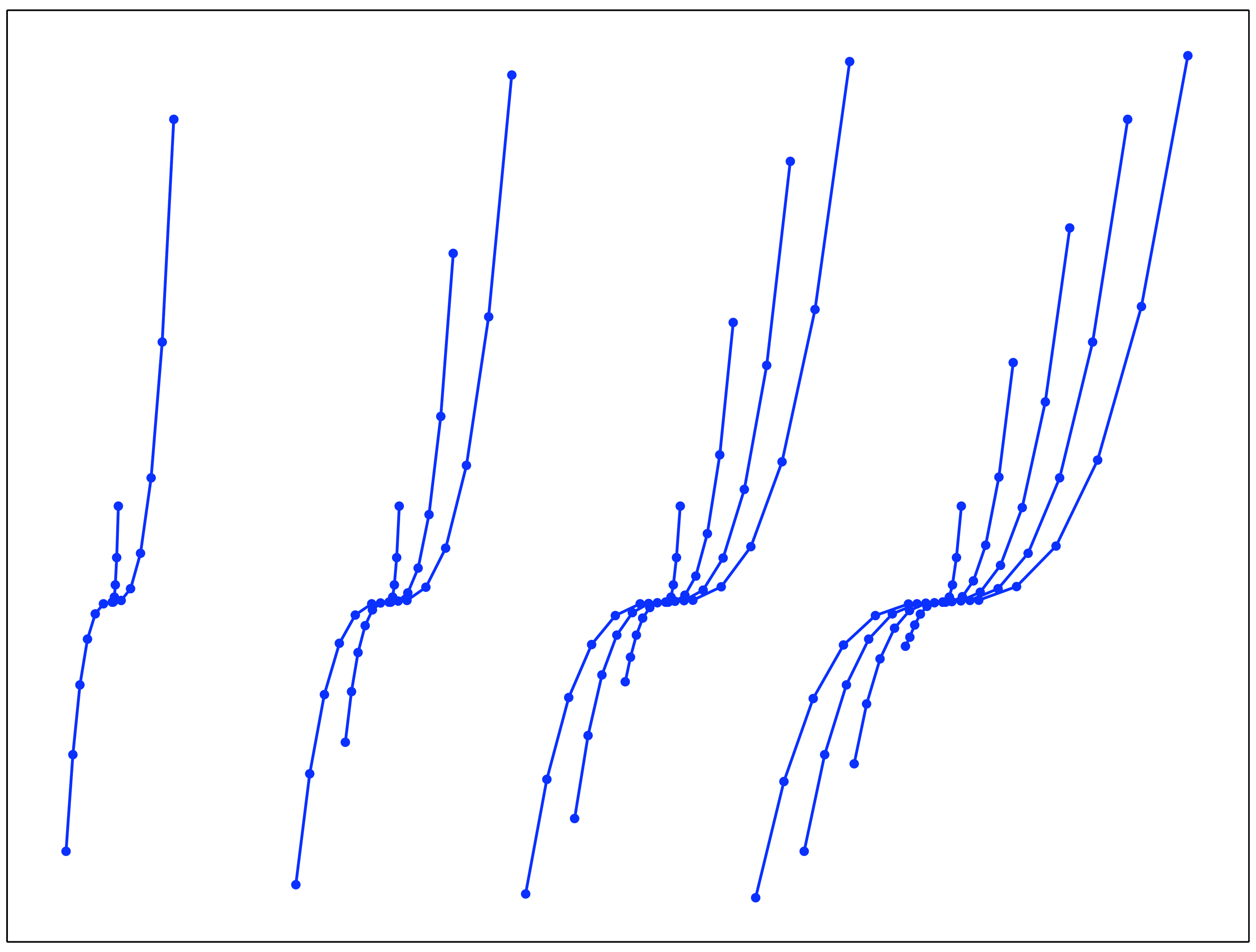
For $a = 0$, the metric is spherically symmetric, and (4) is solved by spherical harmonics $\lambda = l(l+1)$, $l \geq |k|$; the QNM's with $\operatorname{Im}\omega = O(1)$ and $\operatorname{Re}\omega \gg 0$ asymptotically lie on a lattice [SZ]:

$$\omega \sim \frac{\sqrt{1 - 9\Lambda M^2}}{3\sqrt{3}M}[\pm(l + 1/2) - i(m + 1/2)]. \qquad (5)$$

Here l and m are nonnegative integers. The QNM in (5) has multiplicity $2l + 1$ as it corresponds to $k = -l, \dots, l$. In the case $a \neq 0$, however, the spherical symmetry breaks down and we observe an analogue of the Zeeman effect: each QNM in (5) splits into $2l + 1$ QNM's; each of them corresponds to its own value of k.

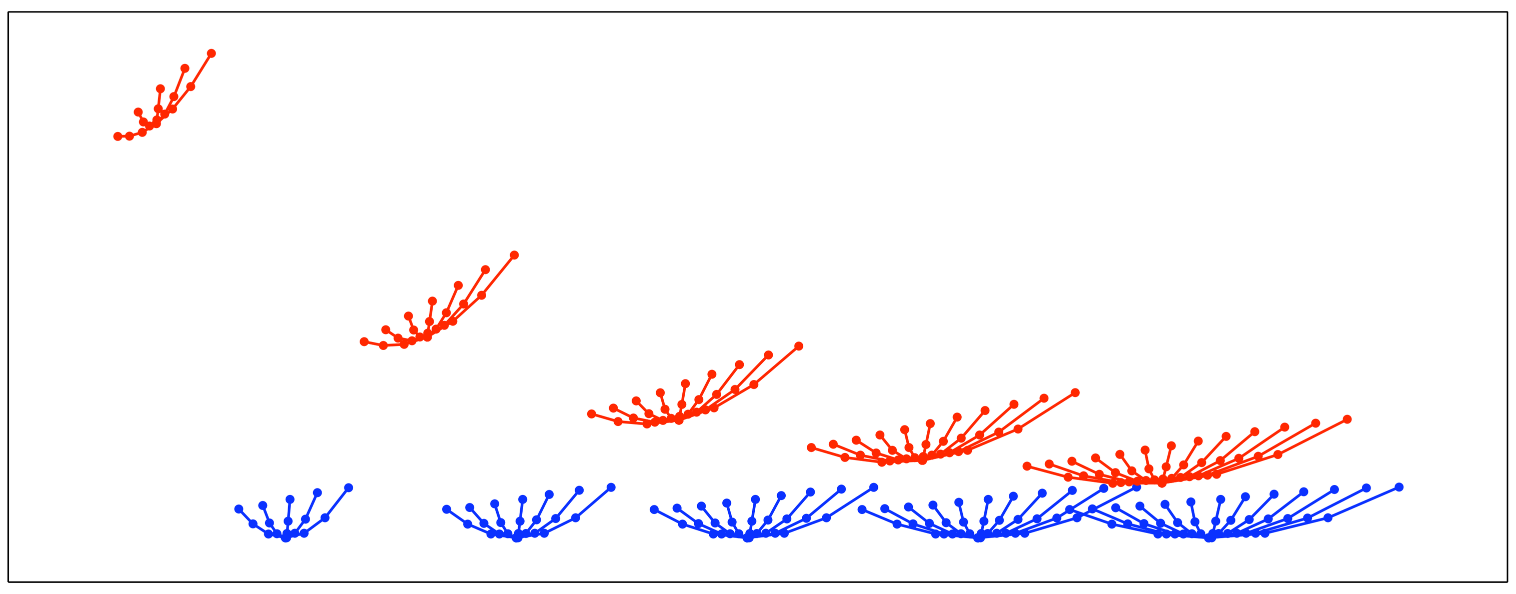


First degree approximation of QNM's for $M = 1, \Lambda = 0.05, a = 0.15, l = 1, 2, 3, m = 0, 1.$



First degree approximation of QNM's for $\Lambda = 0.05$, $l = 1, 2, 3, 4$, $m = 0$, $\operatorname{Re}\omega > 0$. *Each line displays the QNM's for k fixed and* $a = 0, 0.05, \dots, 0.3$.

This effect has previously been observed by physicists in numerical experiments:



Blue: first degree approximation to QNM's for $\Lambda = 0$, $l = 2, \dots, 6$, $m = 0$, $a = 0, 0.1, 0.2, 0.3$. *Red: QNM's as computed in [BCS]. We see that the approximation gets better as* $\operatorname{Re}\omega$ *increases.*

In scattering theory, resonances with $\operatorname{Im}\omega = O(1)$ are typically generated by trapping. The trapping in our situation is normally hyperbolic, with the trapped set having codimension 2. For $a = 0$, it comes from the photon sphere $\{r = 3M\}$, consisting of trapped light rays. The situation for $a \neq 0$ is more complicated, but the geodesic flow is still completely integrable.