Notes and references for the Dartmouth Spectral Geometry 2010 minicourse:
Spectral Theory of Hyperbolic Surfaces

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These notes are still somewhat sketchy, but hopefully give some indication where to start looking for background. I apologize for any omissions.

p. 5: Katok [17] is a good concise source for this material. Ratcliffe [26] gives a more encyclopedic treatment, with excellent historical notes.

p. 12: Terras [34] covers a lot of the basic material on automorphic forms.

p. 13: There are many places to find background on uniformization. See e.g. Abikoff [1], Farkas-Kra [10], Jost [16], Petersen [24].

p. 16: See Katok [17] for the proofs.

p. 18: In higher dimensions, geometrically finite and finitely generated are no longer equivalent; see Bowditch [6].

p. 21: See Borthwick [3] for a proof that funnels and cusps are the only possibilities for ends of hyperbolic surfaces.

p. 26: In the compact case, the spectral decomposition can be established using only some basic facts about the heat kernel; see Buser [7].

p. 27: Stone's formula is a fairly direct consequence of the spectral theorem. Reed-Simon [27] is a standard reference for this material.

p. 28: The derivation of this formula for $R_{\mathbb{H}}(s)$ is covered thoroughly in Borthwick [3].

p. 29: Faddeev [9] was the first to prove meromorphic continuation of the resolvent in this context.

p. 30: This method was introduced by Guillopé-Zworski [12], following the philosophy of Mazzeo-Melrose [21]. Older methods relied more on the group structure. The advantage here is that a compactly supported perturbation, whether metric or potential, can be accommodated without change to the proof.

p. 31: (1) is a general statement for any compact Riemannian manifold. (2) was first proven by Selberg. For proofs and discussion in this case see Lax-Phillips [20], Venkov [35]. (3) The spectral picture for the infinite-area case was first worked out by Lax-Phillips [19]. An expositiory treatment of this case is given in Borthwick [3].

p. 32: This is the resonance plot for scattering by a circular obstacle in $\mathbb{H}$, taken from Borthwick [4].

p. 33: The following material was adapted from Apostol [2] and Venkov [35].

p. 35: See Phillips-Sarnak [25].

p. 36: This discussion of Eisenstein series follows Venkov [35]. Selberg proved this in 1950; his proof is described in the Göttingen lectures [32]. Deriving meromorphic continuation from the resolvent is Faddeev's approach [9].

p. 38: The original source is Selberg [31]. Faddeev [9] gave a proof more in the spirit of scattering theory.

p. 39: $\varphi(s)$ is sometimes called the intertwining function.

p. 40: This is the resonance plot for the modular surface. The cusp from data was taken from Hejhal [13].

p. 42: This material is treated in detail in Borthwick [3].

p. 43: This treatment of the Selberg Trace Formula essentially follows the treatment of Buser [7]. There is a nice concise treatment in McKean [22]. (And of course there are many other places to find this.)

p. 53: For the full formula see e.g. Venkov [35].

p. 56: In the finite-area case there is in fact a functional relation, relating $Z(s)$ to $Z(1-s)$, involving the scattering determinant. This was generalized to the infinite-area case in Borthwick-Judge-Perry [5].

p. 57: (1) This is Huber's theorem [14] (see McKean [22] for a straightforward derivation from the heat trace). It was generalized to non-compact finite-area surfaces by Müller [23] and to infinite-area surfaces by Borthwick-Judge-Perry [5]. In all these cases, a consequence is that the resonance set determines the surface up to finitely many possibilities. (2) Huber [14] proved this in the compact case. The non-compact finite volume case is due to Sarnak [28]. The infinite-area case was proven independently by Guillopé [11] and Lalley [18]. (3) This extension of the Weyl law is due to Selberg [32]. There is no analog in the infinite-area case (at least not yet).

p. 58: Katok [17] gives a nice introduction to arithmetic Fuchsian groups.

p. 64: This formula was worked out by Selberg [32].

p. 70: The proof given here comes from Apostol [2].

p. 75: Before defining $L(s, \phi)$, I should have mentioned that Hecke operators satisfy

$$T_n T_m = \sum_{d|(n,m)} T_{mn/d^2}.$$

p. 76: For more details see the review article by Sarnak [29].

p. 78: For quantum ergodicity, the references are Šnirelman [33], Zelditch [36], and Colin de Verdière [8]. The very recent history of arithmetic QUE is summarized by Sarnak [30].

p. 79: This by no means an exhaustive list - these are the main references I was looking at while preparing the lectures. The books of Iwaniec [15] and Terras [34] are also good background sources.

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