Combinatorial properties of triangular partitions

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Dartmouth College

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Outline

1. Background
2. Characterizations of triangular partitions
3. The triangular Young poset
4. Bijections and efficient generation
5. Generating functions for subsets of triangular partitions
6. Triangular partitions inside a rectangle

1. Background

Partitions

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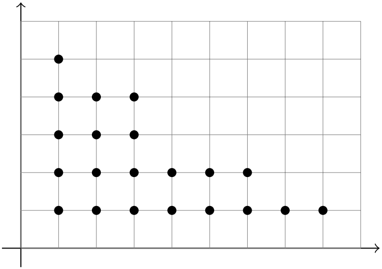
The *Ferrers diagram* of λ is the set of lattice points

$$\{(a,b) \in \mathbb{N}^2 \mid 1 \leq b \leq k,\ 1 \leq a \leq \lambda_b\}.$$

$\lambda = 86331$



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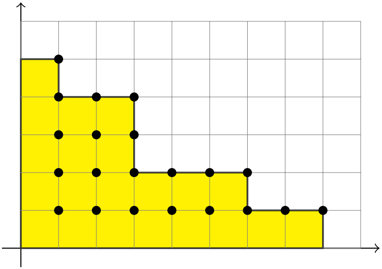


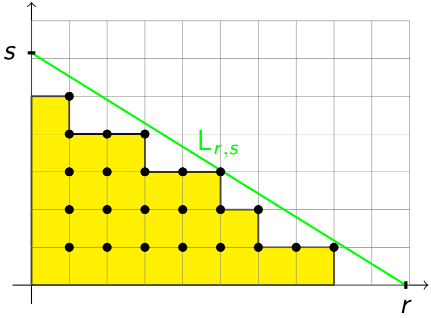
The *Young diagram* of λ is the set of unit squares (called *cells*) whose north-east corners are the points in the Ferrers diagram.

Triangular partitions

Definition

A partition is *triangular* if its Ferrers diagram consists of the points in $\mathbb{N}^2$ that lie on or below a line (called a *cutting line*).

$\tau = 86531$



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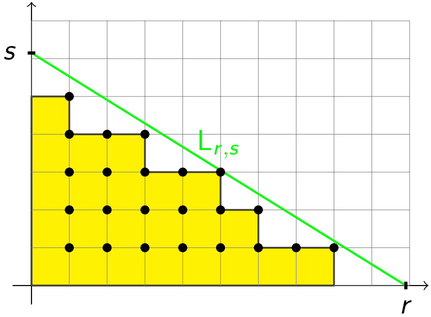

$\Delta(n) =$ set of triangular partitions of n

$$\Delta = \bigcup_{n\geq 0} \Delta(n)$$

History of triangular partitions

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- In 2023, Bergeron and Mazin coined the term *triangular partitions* and studied some of their combinatorial properties.

2. Characterizations of triangular partitions

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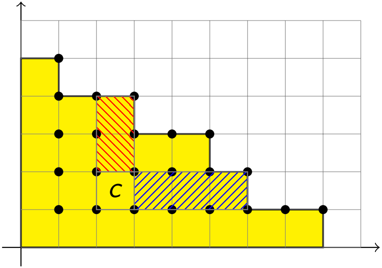
Proposition (Bergeron, Mazin '23)

A partition λ is triangular if and only if

$$\max_{c\in\lambda} \frac{\operatorname{leg}(c)}{\operatorname{arm}(c)+\operatorname{leg}(c)+1} < \min_{c\in\lambda} \frac{\operatorname{leg}(c)+1}{\operatorname{arm}(c)+\operatorname{leg}(c)+1}.$$



$\operatorname{leg}(c) = 2$

$\operatorname{arm}(c) = 3$

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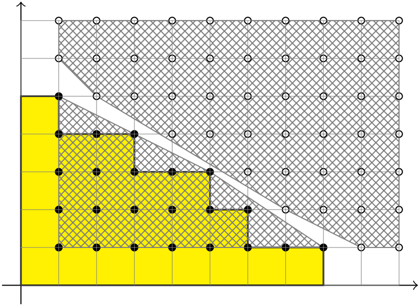

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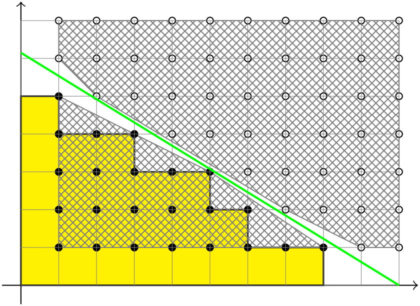
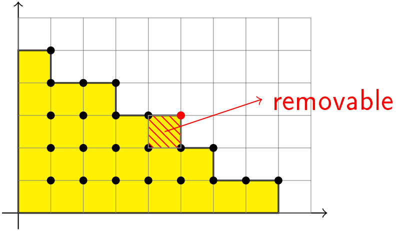


Addable and removable cells

Definition

A cell of a triangular partition τ is *removable* if removing it from τ yields a triangular partition.

$\tau = 86531$



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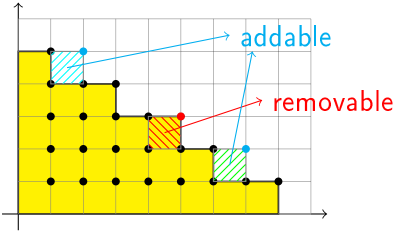

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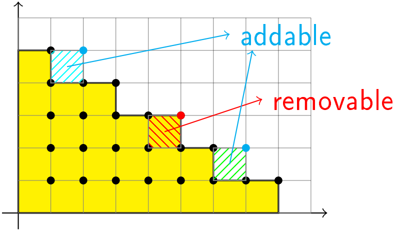

Lemma (Bergeron, Mazin '23)

A nonempty triangular partition can have: one removable cell and two addable cells, two removable cells and one addable cell, or two removable cells and two addable cells.

Finding removable and addable cells

Proposition (E., Galván '23)

Two cells in a triangular partition τ are removable if and only if:

- they are consecutive vertices of $\mathrm{Conv}(\tau)$, and
- the line passing through them does not intersect $\mathrm{Conv}(\mathbb{N}^2 \setminus \tau)$.

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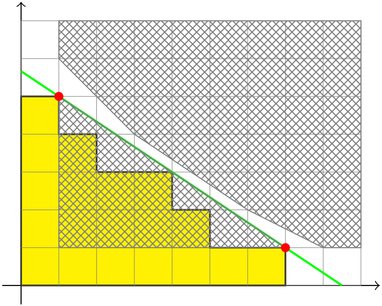

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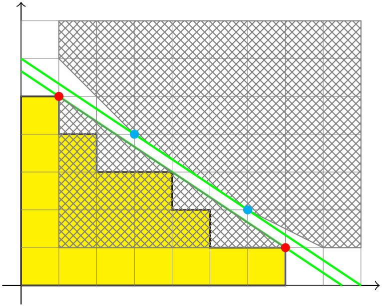

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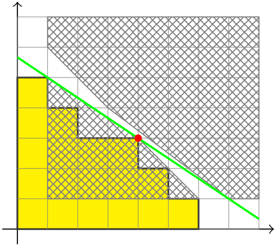
Finding removable and addable cells

Proposition (E., Galván '23)

A cell c in a triangular partition τ is its only removable cell if and only if:

- c is a vertex of $\mathrm{Conv}(\tau)$,
- the line extending the edge of $\mathrm{Conv}(\tau)$ adjacent to c from the left intersects $\mathrm{Conv}(\mathbb{N}^2 \setminus \tau)$ to the right of c,



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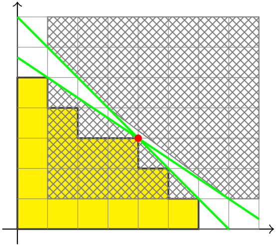

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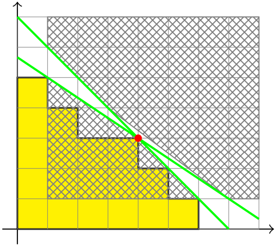
There is an analogous characterization for a single addable cells.



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An algorithm to determine triangularity

Proposition (E., Galván '23)

Let $\lambda \vdash n$ with k parts. Using the above characterization, we can determine whether λ is triangular (and if so, find its addable and removable cells) in time $\mathcal{O}(k)$.

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Sketch of the algorithm:

1. Use Graham's scan to find the vertices of $\mathrm{Conv}(\lambda)$ and $\mathrm{Conv}(\mathbb{N}^2 \setminus \lambda)$.
2. Perform a binary search on the boundary of $\mathrm{Conv}(\lambda)$ to look for a pair of removable cells.
   For each edge, finding a point in $\mathbb{N}^2 \setminus \lambda$ that lies below the line extending the edge tells us in which direction to keep searching.
3. If no pair of removable cells is found, apply the same procedure to the boundary of $\mathrm{Conv}(\mathbb{N}^2 \setminus \lambda)$ to find a pair of addable cells.

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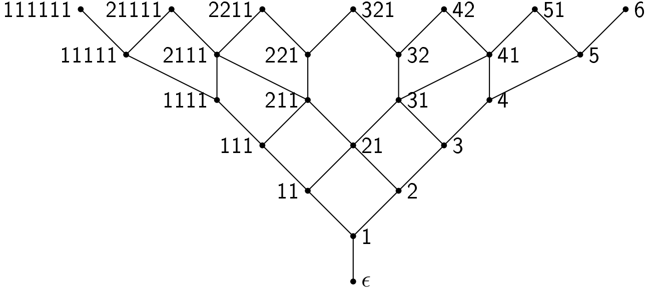
For comparison, an algorithm based on the characterization of Bergeron–Mazin would take time $\mathcal{O}(n)$ just to determine triangularity.

3. The triangular Young poset

The triangular Young poset

Bergeron and Mazin considered the poset $\mathbb{Y}_\Delta$ of triangular partitions ordered by containment of their Young diagrams:



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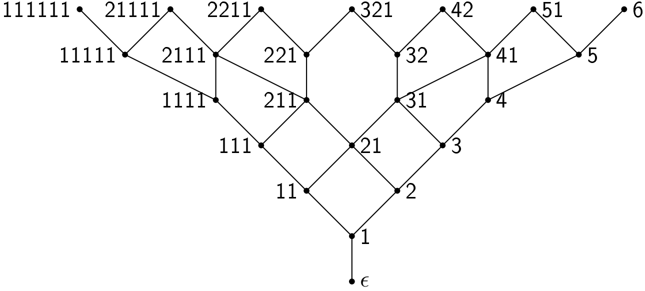

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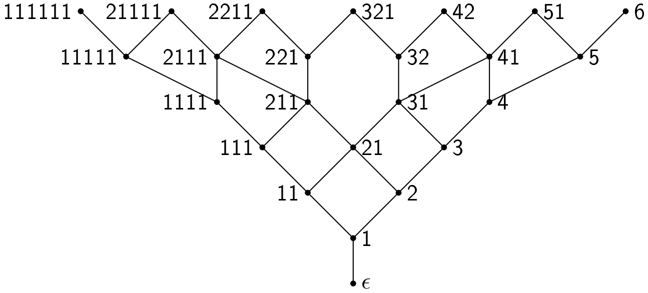

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In particular, $\mathbb{Y}_\Delta$ is ranked by the size of the partitions.

Properties of the triangular Young poset

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The poset $\mathbb{Y}_\Delta$ has a planar Hasse diagram, and it is a lattice.

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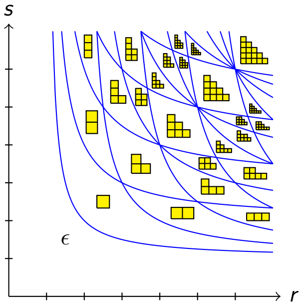
To prove this, they define a moduli space of lines, where

- each point (r, s) represents the line $\mathsf{L}_{r,s}$,
- the lines through a lattice point are represented by a hyperbola,
- the resulting regions are in bijection with triangular partitions.



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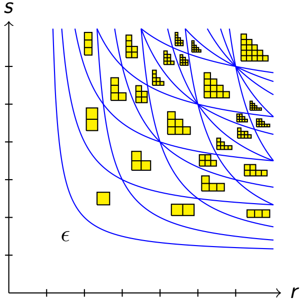

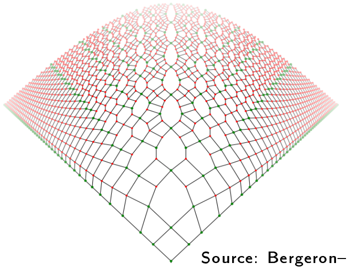

Source: Bergeron–Mazin

The join and the meet in $\mathbb{Y}_\Delta$

Definition

A poset is a *lattice* if every pair of elements τ and ν has:

- a least upper bound, denoted by $\tau \vee \nu$ (called the join), and
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Proposition (E., Galván '23)

For any $\tau, \nu \in \mathbb{Y}_\Delta$,

$$\tau \vee \nu = \mathbb{N}^2 \cap \mathrm{Conv}(\tau \cup \nu),$$
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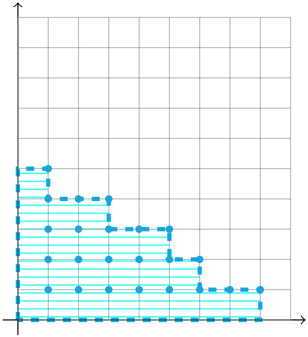
Example: 86531 .



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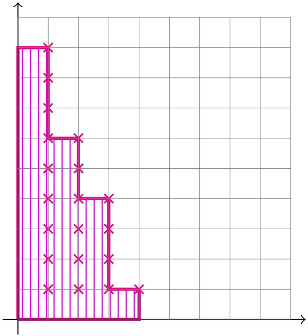

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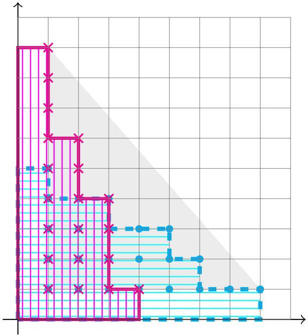

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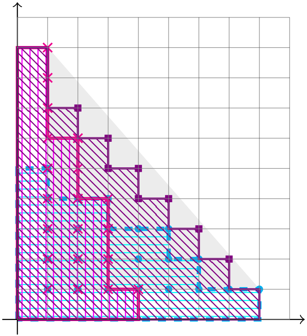

The Möbius function of the poset $\mathbb{Y}_\Delta$

Denote the Möbius function of $\mathbb{Y}_\Delta$ by μ.

Theorem (E., Galván '23)

Let $\tau, \nu \in \mathbb{Y}_\Delta$ such that $\tau \leq \nu$. Then

$$\mu(\tau,\nu) = \begin{cases} 1 & \textit{if either } \tau = \nu \textit{ or} \\ & \textit{there exist } \zeta^1 \neq \zeta^2 \textit{ such that } \tau \lessdot \zeta^1, \zeta^2 \textit{ and } \nu = \zeta^1 \vee \zeta^2, \\ -1 & \textit{if } \tau \lessdot \nu, \\ 0 & \textit{otherwise.} \end{cases}$$

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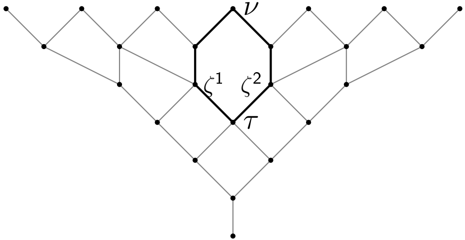

$\mu(\tau,\nu) = 1$

4. Encodings as balanced words and efficient generation

Balanced words

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for any $h \leq \ell$ and $i, j \leq \ell - h + 1$.

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Let $\mathcal{B}$ be the set of all balanced words, and $\mathcal{B}_\ell$ the set of those of length ℓ.

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Balanced words can also be defined as factors of *Sturmian words*.

Let $\mathcal{B}$ be the set of all balanced words, and $\mathcal{B}_\ell$ the set of those of length ℓ.

Theorem (Lipatov '82)

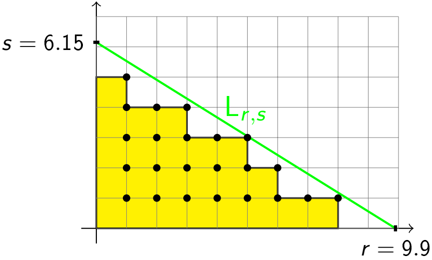
$$|\mathcal{B}_\ell| = 1 + \sum_{i=1}^{\ell} (\ell - i + 1)\varphi(i),$$

where φ denotes Euler's totient function.

Wide triangular partitions

Definition

A triangular partition is *wide* if it has a cutting line $L_{r,s}$ with $r > s$.
Δ_{wide} = set of wide triangular partitions.



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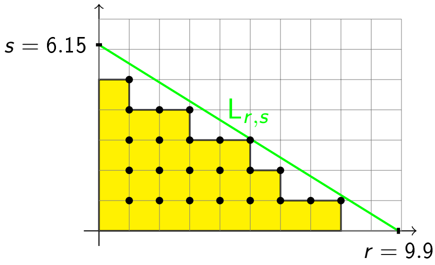

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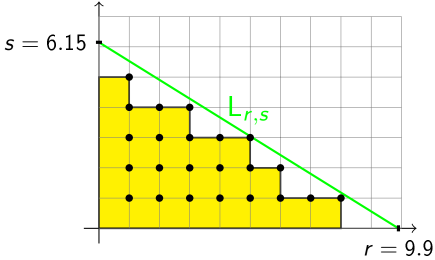

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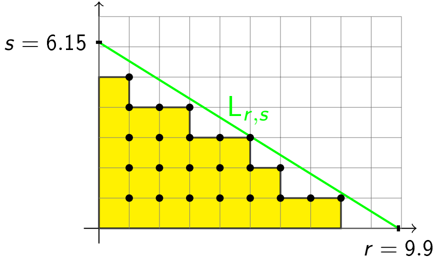

Lemma

- *A triangular partition is wide if and only if its parts are distinct.*
- *For every triangular partition τ, either τ or its conjugate are wide. Both are wide if and only if $\tau = k(k-1)\dots 21$ for some k (staircase).*

First encoding of triangular partitions as balanced words

Given $\tau = \tau_1 \dots \tau_k \in \Delta_{\mathsf{wide}}$, define

$$\omega(\tau) = 10^{\tau_1 - \tau_2 - 1} 10^{\tau_2 - \tau_3 - 1} \dots 10^{\tau_{k-1} - \tau_k - 1} 10^{\tau_k - 1}.$$

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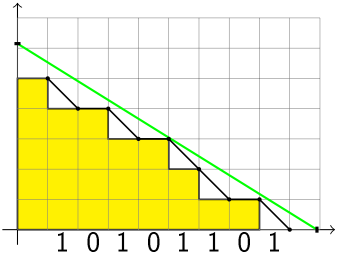

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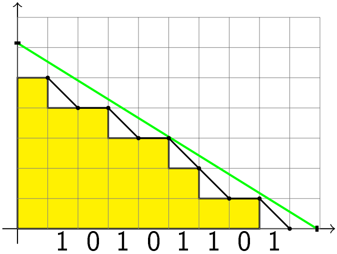

Proposition (E., Galván '23)

For every $k, \ell \geq 1$, the map ω is a bijection

$$\{\tau = \tau_1 \dots \tau_k \in \Delta_{\text{wide}} \mid \tau_1 = \ell\} \longrightarrow \{w = w_1 \dots w_\ell \in \mathcal{B}_\ell \mid w \text{ has } k \text{ ones and } w_1 = 1\}.$$

Second encoding of triangular partitions as balanced words

For $\tau = \tau_1 \dots \tau_k \in \Delta_{\text{wide}}$ with $k \geq 2$, let

$$\begin{aligned}
\min(\tau) &= \tau_k, \\
\mathcal{D}(\tau) &= \{\tau_1 - \tau_2,\ \tau_2 - \tau_3, \dots, \tau_{k-1} - \tau_k\}, \\
\operatorname{dif}(\tau) &= \min \mathcal{D}(\tau), \\
\operatorname{wrd}(\tau) &= w_1 \dots w_{k-1}, \quad \text{where } w_i = \tau_i - \tau_{i+1} - \operatorname{dif}(\tau) \text{ for all } i.
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Example: $\tau = (12, 9, 7, 4, 1)$

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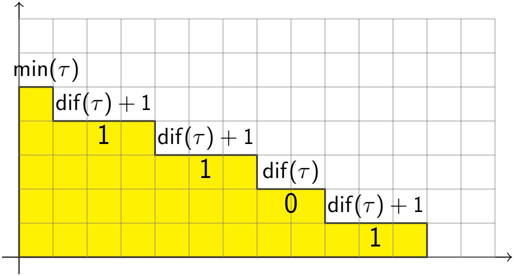

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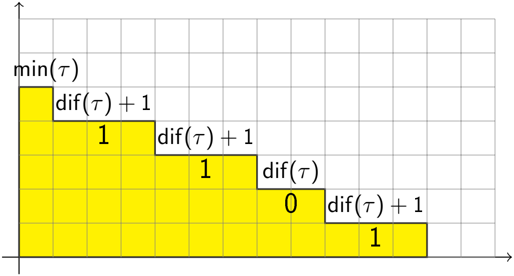

Second encoding and efficient generation

Let $\mathcal{B}^0$ denote the set of balanced words that contain at least one zero.

Theorem (E., Galván '23)

The map $\chi = (\mathrm{min}, \mathrm{dif}, \mathrm{wrd})$ *is a bijection between the set of wide triangular partitions with at least two parts and the set*

$$\mathcal{T} = \{(m, d, w) \in \mathbb{N} \times \mathbb{N} \times \mathcal{B}^0 \mid m \leq d + 1;\ w1 \in \mathcal{B}^0 \text{ if } m = d + 1\}.$$

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Theorem (E., Galván '23)

There is an algorithm that finds $|\Delta(n)|$ *for* $1 \leq n \leq N$ *in time* $\mathcal{O}(N^{5/2})$.

1. Perform a depth first search through the tree of balanced words of length $\leq \lfloor \sqrt{2N} \rfloor$. The children of a word w can be $w0$ and/or $w1$.
2. For each w in the tree, search through the pairs (m, d) such that $(m, d, w) \in \mathcal{T}$ and the size of the corresponding partition is $\leq N$.
3. Each triplet (m, d, w) accounts for two triangular partitions (conjugate of each other), unless it corresponds to the staircase partition.

The sequence $|\Delta(n)|$

This algorithm allows us to compute the first 10^5 terms of the sequence $|\Delta(n)|$, compared to the 39 terms that had been previously computed.

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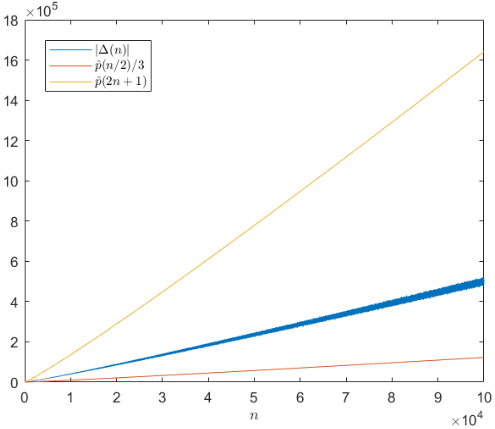
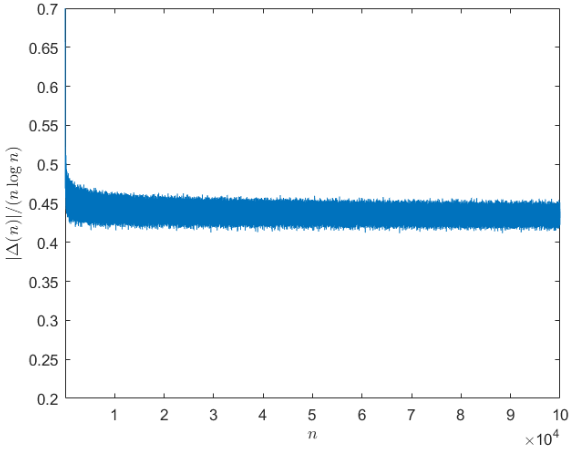


The sequence $|\Delta(n)|$ and the bounds $C\, n \log n < |\Delta(n)| < C' n \log n$ given by Corteel–Rémond–Schaeffer–Thomas '99.

The sequence $|\Delta(n)|/(n \log n)$



The sequence $|\Delta(n)|/(n \log n)$ seems to oscillate between 0.42 and 0.45.

5. Generating functions

Generating functions for (subsets of) triangular partitions

Theorem (Corteel, Rémond, Schaeffer, Thomas '99)

$$\sum_{n\geq 0}|\Delta(n)|z^n = \frac{1}{1-z} + \sum_{\gcd(a,b)=1}\ \sum_{\substack{0\leq j<a\\ 0\leq i<b}}\ \sum_{1\leq m<k} z^{N_\Delta(a,b,k,m,i,j)},$$

where

$$N_\Delta(a,b,k,m,i,j) = (k-1)\left(\frac{(a+1)(b+1)}{2}-1\right) + \binom{k-1}{2}ab + ij$$
$$+\, i(k-1)a + j(k-1)b + T(a,b,j) + T(b,a,i) + m$$

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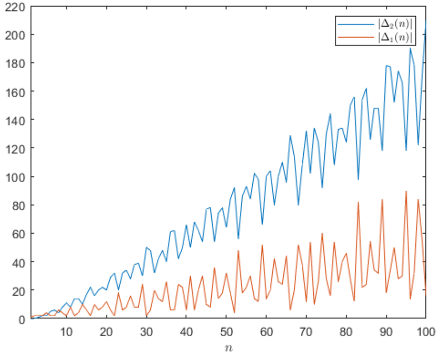
We can give similar generating functions for partitions with a given number (i.e. one or two) of removable and addable cells.

One removable vs two removable cells

Let $\Delta_1(n), \Delta_2(n) \subset \Delta(n)$ denote the subsets of partitions with one and two removable cells, respectively.



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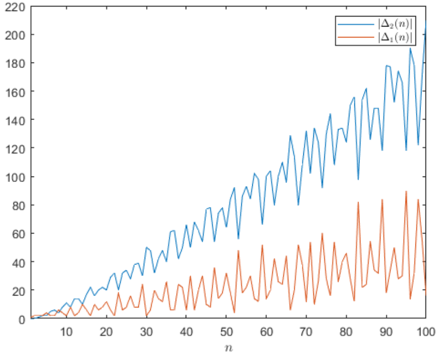

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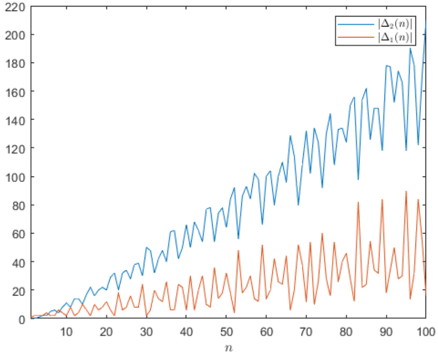

Open questions:

- Is $|\Delta_2(n)| > |\Delta_1(n)|$ for all $n \geq 9$?
- Do the local maxima of $|\Delta_1(n)|$ and the local minima of $|\Delta_2(n)|$ always occur when $n \equiv 2 \pmod 3$?

6. Triangular partitions inside a rectangle

Triangular partitions inside a square

$\Delta^{h\times\ell}$ = set of triangular partitions whose Young diagram fits inside an $h \times \ell$ rectangle (i.e., with $\leq h$ parts and largest part $\leq \ell$).

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Proof idea:

- Use our first encoding as balanced words.
- Apply Lipatov's enumeration formula for balanced words.

Triangular partitions inside a square

We can also give a direct combinatorial proof:

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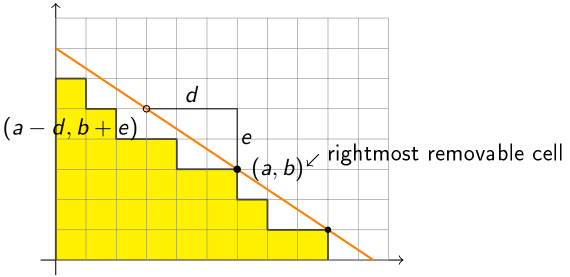
- Construct a bijection between triangular partitions and

$$Q = \{(a, b, d, e) \in \mathbb{N}^4 \mid d < a,\ \gcd(d, e) = 1\}.$$



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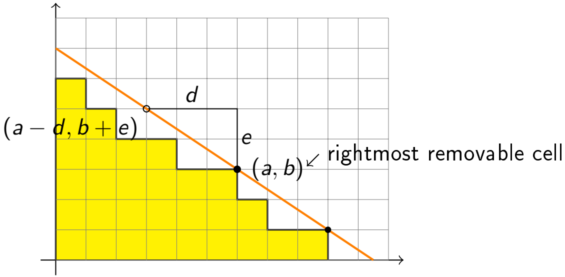

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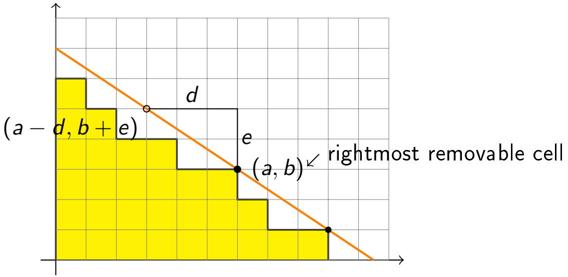

- Characterize the tuples (a, b, d, e) coming from partitions in $\Delta^{\ell\times\ell}$.
- For fixed $d < e$ with $\gcd(d, e) = 1$, the tuples of the form (a, b, d, e) and $(a, b, e, e-d)$ are in bijection with the lattice points inside a certain triangle, which are counted by $\binom{\ell-e+2}{2}$.

Triangular partitions inside a rectangle

The above argument also gives a new combinatorial proof of Lipatov's enumeration formula for balanced words.

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We have similar formulas for other rectangles:

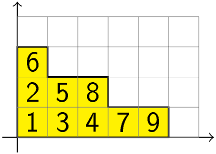
$$\left|\Delta^{\ell\times(\ell-1)}\right| = \frac{1}{2} + \frac{1}{2}\sum_{i=1}^{\ell}(\ell-i+1)^2\varphi(i),$$

$$\left|\Delta^{\ell\times(\ell-2)}\right| = 1-\ell+\sum_{i=1}^{\ell}\left(\binom{\ell-i+1}{2}+\frac{1}{2}\right)\varphi(i).$$

But not for the general case $\left|\Delta^{h\times\ell}\right|$.

Further research

- Triangular Young tableaux.



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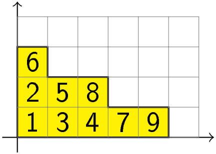

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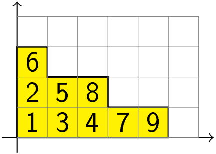

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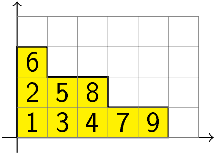

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Thank you!