Bijections for lattice paths between two boundaries

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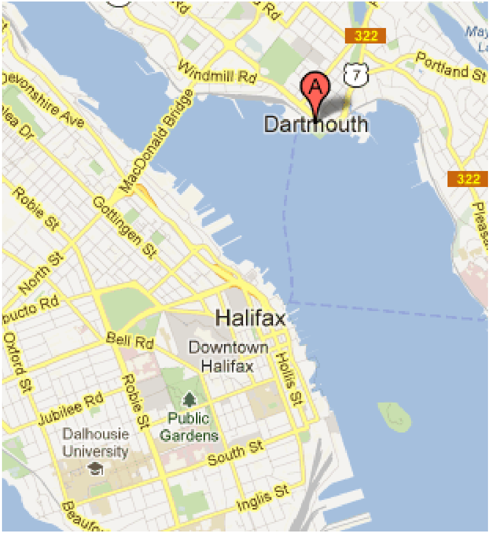
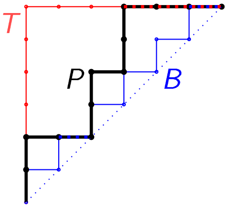
Joint work with Martin Rubey

A different Dartmouth



Dyck paths



$t(P) = 3$ $b(P) = 2$

For $P \in \mathcal{D}_n$ (Dyck paths with $2n$ steps), let

$$t(P) = \#\text{ of } E \text{ steps in common with } T$$
$$= \text{“height” of the last “peak”}$$
$$b(P) = \#\text{ of } E \text{ steps in common with } B$$
$$= \text{number of returns}$$

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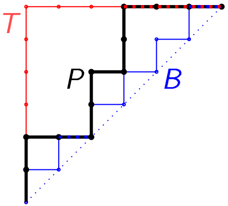

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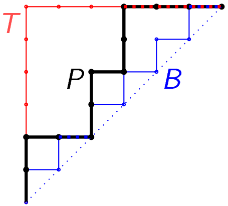
Theorem (Deutsch '98)

The joint distribution of the pair (t, b) over $\mathcal{D}_n$ is symmetric, i.e.,

$$\sum_{P \in \mathcal{D}_n} x^{t(P)} y^{b(P)} = \sum_{P \in \mathcal{D}_n} x^{b(P)} y^{t(P)}.$$

Dyck paths



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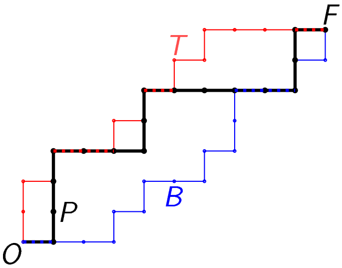
Proof 1 (Deutsch): Recursive bijection. Proof 2: Generating fcts.
Both proofs rely on the recursive structure of Dyck paths.

A generalization to arbitrary boundaries



$t(P) = 4 \quad b(P) = 3$

T and B paths from O to F with steps N and E, with T weakly above B

$P \in \mathcal{P}(T, B) =$ set of paths from O to F weakly between T and B

$t(P) = \#$ of E steps in common with T (top contacts of P)

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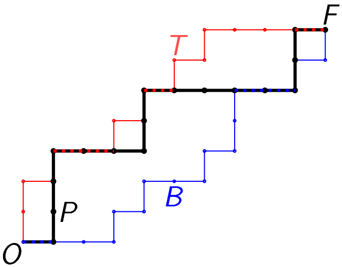

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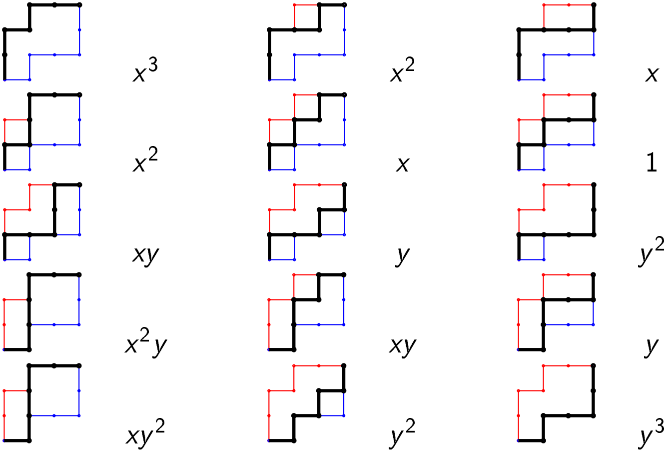
The joint distribution of (t, b) over $\mathcal{P}(T, B)$ is symmetric, i.e.,

$$\sum_{P \in \mathcal{P}(T,B)} x^{t(P)} y^{b(P)} = \sum_{P \in \mathcal{P}(T,B)} x^{b(P)} y^{t(P)}.$$

Example



$$\sum_{P\in\mathcal{P}(T,B)} x^{t(P)}y^{b(P)} = x^3+x^2y+xy^2+y^3+2x^2+2xy+2y^2+2x+2y+1$$

Proof

The known proofs for Dyck paths do not seem to generalize to arbitrary boundaries.

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We give an involution

$$\Phi : \mathcal{P}(T, B) \to \mathcal{P}(T, B)$$

with the property $t(\Phi(P)) = b(P)$ and $b(\Phi(P)) = t(P)$.

Idea: Given $P \in \mathcal{P}(T, B)$ with $t(P) > b(P)$, turn some of its top contacts into bottom contacts, one at a time.

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Idea: Given $P \in \mathcal{P}(T, B)$ with $t(P) > b(P)$, turn some of its top contacts into bottom contacts, one at a time.

Which ones? How?

Proof – warm up

A transformation on words

Given a $\mathbf{w}$ word over the alphabet $\{\mathbf{t}, \mathbf{b}\}$, define $\mu(\mathbf{w})$ as follows:

- Draw a path with a step $(1, 1)$ for each $\mathbf{t}$, and a step $(1, -1)$ for each $\mathbf{b}$.

$\mathbf{w} = \mathbf{bttbtbbbttbttbtbtt}$

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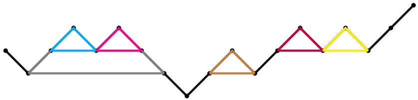

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- Match $\mathbf{t}$'s and $\mathbf{b}$'s that “face” each other in the path.
- Replace the leftmost unmatched $\mathbf{t}$ with a $\mathbf{b}$.
  (If no unmatched $\mathbf{t}$, then $\mu(\mathbf{w})$ is not defined.)

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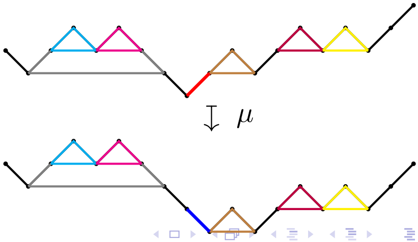

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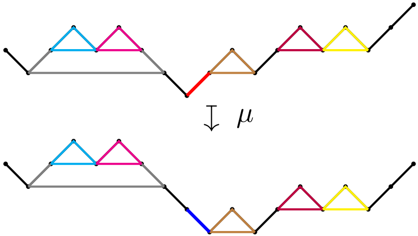
A transformation on words

$\mathbf{w} = $ **bttbtbbbttbttbtbtt**

$\mu(\mathbf{w}) = $ **bttbtbbbbtbttbtbtt**



Note: $\mathbf{w}$ can be recovered from $\mu(\mathbf{w})$ by replacing the rightmost unmatched **b** with a **t**.

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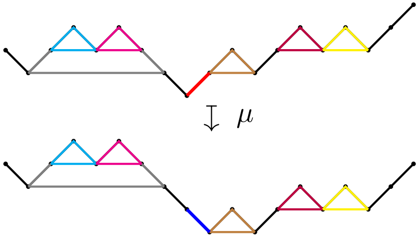

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Lemma

μ^{e-f} is a bijection between

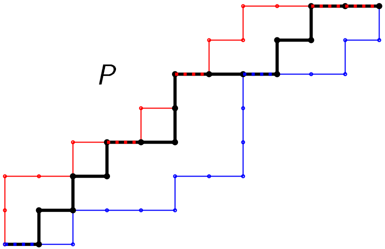
- *words with e **t**'s and f **b**'s, and*
- *words with f **t**'s and e **b**'s.*

Proof – the bijection

A transformation on paths

Given $P \in \mathcal{P}(T, B)$, define $\phi(P)$ as follows:

- Record top and bottom contacts of P as a word $\mathbf{w}$ over $\{\mathbf{t}, \mathbf{b}\}$.



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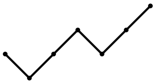

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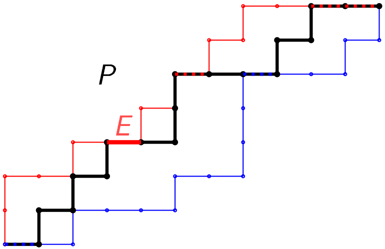

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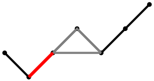

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- Record top and bottom contacts of P as a word $\mathbf{w}$ over $\{\mathbf{t}, \mathbf{b}\}$.
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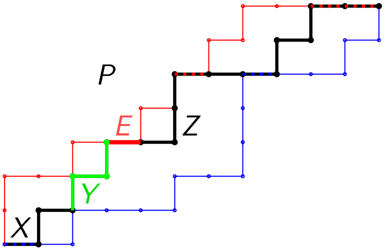
- Write $P = XYEZ$, where Y touches B only at its left endpoint.



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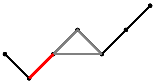

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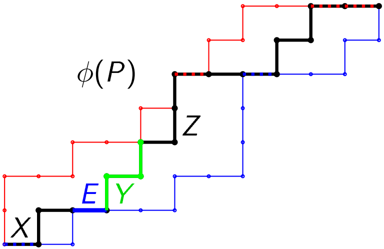
- Write $P = XYEZ$, where Y touches B only at its left endpoint.
- Let $\phi(P) = XEYZ$.



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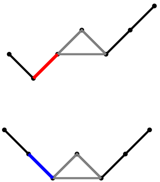

Proof – the bijection

A transformation on paths

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Φ is an involution on $\mathcal{P}(T, B)$ that satisfies $t(\Phi(P)) = b(P)$ and $b(\Phi(P)) = t(P)$.

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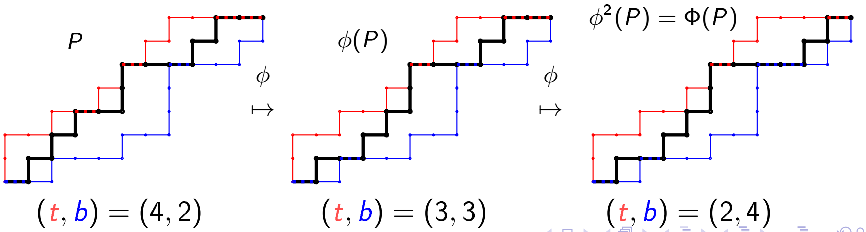
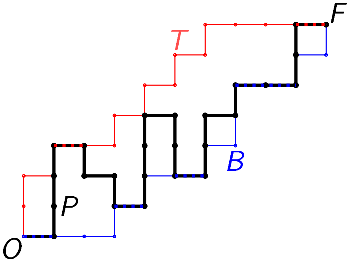


A generalization to paths with S steps



$\widetilde{\mathcal{P}}(T, B)$ = set of paths from O to F with steps N, E and S weakly between T and B.

For $P \in \widetilde{\mathcal{P}}(T, B)$, define $t(P)$ and $b(P)$ as before.
The *descent set* of P is the set of x-coordinates where S steps occur.

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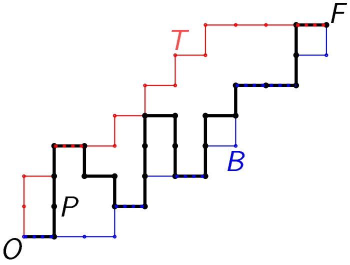

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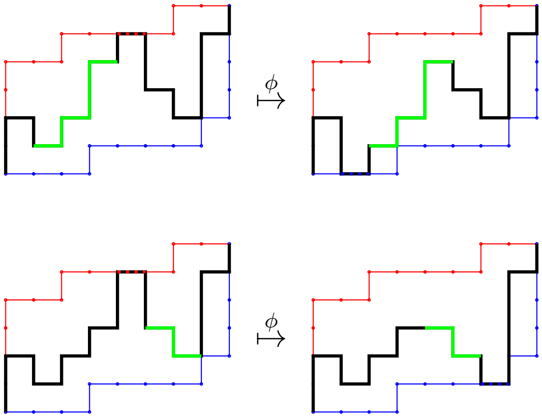
Theorem

There is an involution $\widetilde{\mathcal{P}}(T, B) \to \widetilde{\mathcal{P}}(T, B)$ that switches the statistics (t, b) and preserves the descent set.

A generalization: examples

The map ϕ for paths with S steps:



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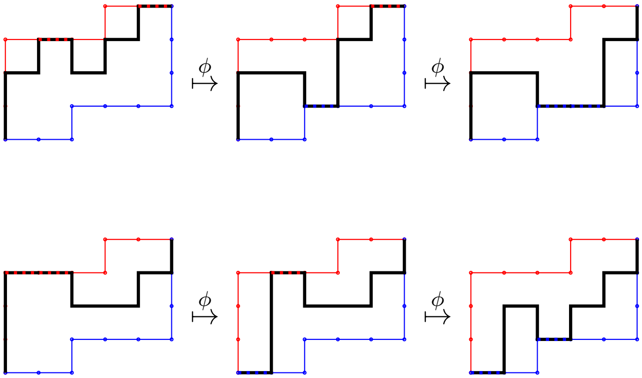
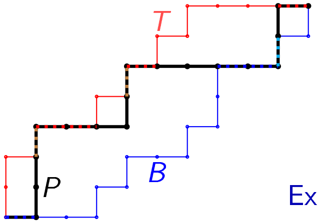


A related theorem



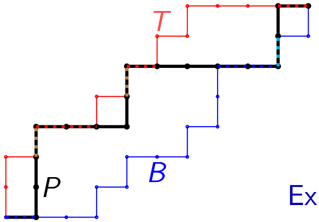
For $P \in \mathcal{P}(T, B)$, let

$\ell(P) = \#$ of N steps in common with T

$r(P) = \#$ of N steps in common with B

Example: $t(P) = 4$, $b(P) = 3$, $\ell(P) = 2$, $r(P) = 1$.

A related theorem



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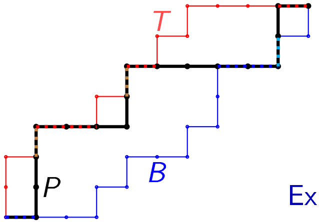
Example: $t(P) = 4$, $b(P) = 3$, $\ell(P) = 2$, $r(P) = 1$.

Theorem

The pairs (b, ℓ) and (t, r) have the same joint distribution over $\mathcal{P}(T, B)$, i.e.,

$$\sum_{P \in \mathcal{P}(T,B)} x^{b(P)} y^{\ell(P)} = \sum_{P \in \mathcal{P}(T,B)} x^{t(P)} y^{r(P)}.$$

A related theorem



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We do not know of a bijective proof similar to the previous one.

Proof idea

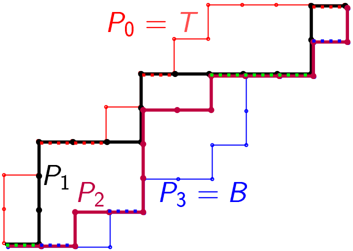
Both

$$\sum_{P\in\mathcal{P}(T,B)} x^{b(P)}y^{\ell(P)} \quad \text{and} \quad \sum_{P\in\mathcal{P}(T,B)} x^{t(P)}y^{r(P)}$$

equal the Tutte polynomial of a *lattice path matroid*, as defined by Bonin–De Mier–Noy '03.

The statistics b and ℓ (t and r) are internal and external activities with respect to different linear orderings of the ground set.

k-fans of paths

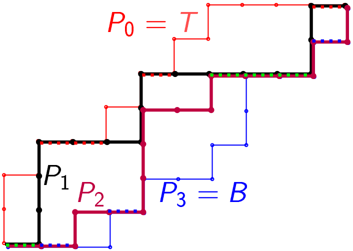


$h_0 = 4 \quad h_1 = 4 \quad h_2 = 6$

$P_1, P_2, \ldots, P_k \in \mathcal{P}(T, B)$,
P_i weakly above P_{i+1} for all i.
Let $P_0 = T$, $P_{k+1} = B$.
For $0 \leq i \leq k$, let

$h_i = \#$ of E steps where
P_i and P_{i+1} conincide

k-fans of paths



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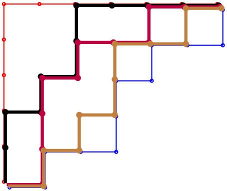
Theorem

The distribution of $(h_0, h_1, \ldots, h_k)$ over k-fans of paths as above is symmetric.

Connection to flagged SSYT

Let $T = NN \dots NEE \dots E$.



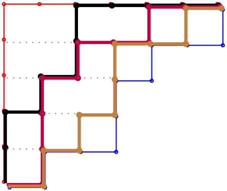
$h_i = \#\ E$ steps in $P_i \cap \mathcal{P}_{i+1}$

$h_0 = 4 \quad h_1 = 3 \quad h_2 = 3 \quad h_3 = 3$

Connection to flagged SSYT

Let $T = NN \dots NEE \dots E$.

$u_1 = 2$
$u_2 = 2$
$u_3 = 1$
$u_4 = 1$



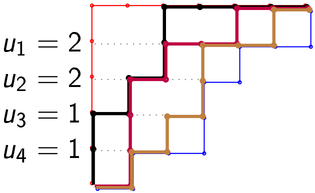
$h_i = \#\ E$ steps in $P_i \cap \mathcal{P}_{i+1}$

$h_0 = 4 \quad h_1 = 3 \quad h_2 = 3 \quad h_3 = 3$

$u_j = \#$ of *unused* E steps at level j

Connection to flagged SSYT

Let $T = NN \dots NEE \dots E$.



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$u_j = \#$ of *unused* E steps at level j

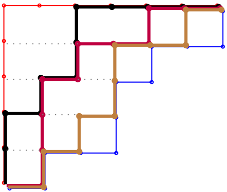
$\lambda = (6, 4, 3, 3, 1)$

T and B form the shape of a Young diagram of a partition λ.

Connection to flagged SSYT

Let $T = NN \ldots NEE \ldots E$.

$u_1 = 2$
$u_2 = 2$
$u_3 = 1$
$u_4 = 1$



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$\lambda = (6, 4, 3, 3, 1)$

T and B form the shape of a Young diagram of a partition λ.

Def: A SSYT of shape λ is called k-**flagged** if the entries in row r are $\leq k + r$ for each r.

1	1	2	2	3	4	≤ 4
2	3	3	4			≤ 5
4	5	6				≤ 6
5	6	7				≤ 7
8						≤ 8

Connection to flagged SSYT

Let $T = NN \dots NEE \dots E$.

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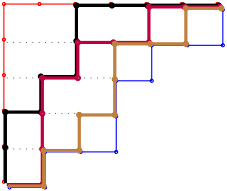
$u_2 = 2$
$u_3 = 1$
$u_4 = 1$



$h_i = \#\ E$ steps in $P_i \cap \mathcal{P}_{i+1}$

$h_0 = 4 \quad h_1 = 3 \quad h_2 = 3 \quad h_3 = 3$

$u_j = \#$ of *unused* E steps at level j

$\lambda = (6, 4, 3, 3, 1)$

T and B form the shape of a Young diagram of a partition λ.

Def: A SSYT of shape λ is called k-**flagged** if the entries in row r are $\leq k + r$ for each r.

1	1	2	2	3	4	≤ 4
2	3	3	4			≤ 5
4	5	6				≤ 6
5	6	7				≤ 7
8						≤ 8

$$\text{weight} = (\#1s, \#2s, \dots) = (2, 3, 3, 3, 2, 2, 1, 1)$$

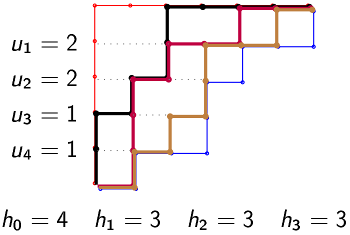
Connection to flagged SSYT

Theorem

There is an explicit bijection between

- *k-fans of paths in $\mathcal{P}(T, B)$ with statistics h_i and u_j, and*
- *k-flagged SSYT of shape λ and weight*

$$(\lambda_1 - h_0, \lambda_1 - h_1, \ldots, \lambda_1 - h_k,\ u_1, u_2, \ldots, u_r).$$



$\mapsto$

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$\lambda_1 = 6$

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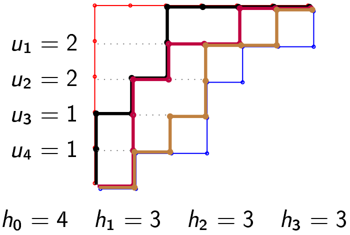

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The bijection uses a variation of *jeu de taquin*.

Connection to k-triangulations

Theorem (E.-Rubey '11, conjectured by C. Nicolás '09)

The joint distribution of the degrees of $k+1$ consecutive vertices in a k-triangulation of a convex n-gon equals the distribution of $(h_0, h_1, \ldots, h_k)$ over k-fans of Dyck paths of semilength $n-2k$.

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The proof uses the previous theorem in the special case of Dyck paths, together with a bijection of Serrano–Stump between k-triangulations and k-flagged SSYT.