Descent sets of cyclic permutations

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The *descent set* of $\pi \in \mathcal{S}_n$ is

$$D(\pi) = \{i \,:\, 1 \le i \le n-1,\ \pi(i) > \pi(i+1)\}.$$

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i is a *weak excedance* of π if $\pi(i) \ge i$.

Allowed patterns of a map

Let X be a linearly ordered set, $f : X \to X$. For each $x \in X$ and $n \geq 1$, consider the sequence

$$x, f(x), f^2(x), \ldots, f^{n-1}(x).$$

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Example

$$\begin{array}{rccc} f : & [0,1] & \to & [0,1] \\ & x & \mapsto & 4x(1-x). \end{array}$$

For $x = 0.8$ and $n = 4$, the sequence

$$0.8,\ 0.64,\ 0.9216,\ 0.2890$$

determines the permutation 3241.

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We'd like to understand the set of forbidden patterns of a given f.

Example

For the map

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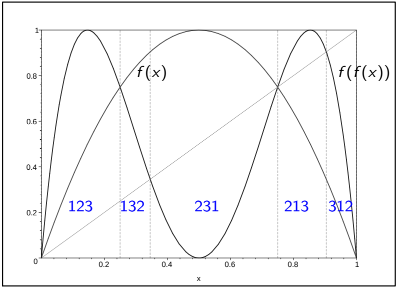

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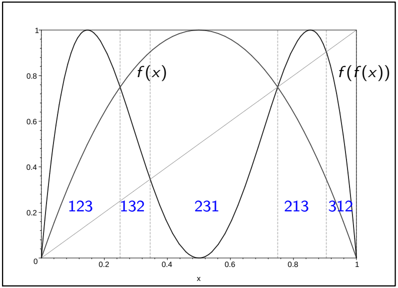

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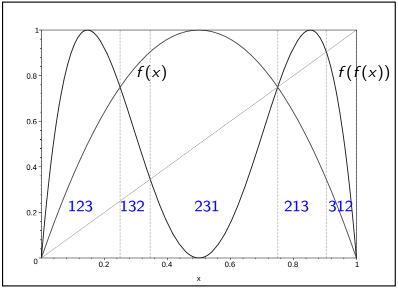

these patterns are forbidden: $321, 1423, 2134, 2143, 3142, 4231, \ldots$

Theorem (E.-Liu): f has infinitely many "basic" forbidden patterns.

Shift maps

For $N \geq 2$, let $\mathcal{W}_N = \{0, 1, \ldots, N-1\}^{\mathbb{N}}$ be the set of infinite words on N letters, equipped with the lexicographic order.

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Define the *shift* on N letters:

$$\begin{array}{rccc} \Sigma_N : & \mathcal{W}_N & \longrightarrow & \mathcal{W}_N \\ & w_1 w_2 w_3 \ldots & \mapsto & w_2 w_3 w_4 \ldots \end{array}$$

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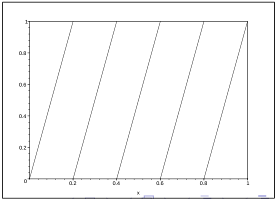
Σ_N has the same allowed/forbidden patterns as the *sawtooth map*

$$\begin{array}{ccc} [0,1] & \rightarrow & [0,1] \\ x & \mapsto & Nx \mod 1 \end{array}$$



Example

The permutation 4217536 is realized (i.e., allowed) by Σ_3, because if $w = 2102212210\ldots \in \mathcal{W}_3$, then

$$\left.\begin{array}{rcl} w = 2102212210\ldots & & 4 \\ \Sigma_3(w) = 102212210\ldots & & 2 \\ {\Sigma_3}^2(w) = 02212210\ldots & & 1 \\ {\Sigma_3}^3(w) = 2212210\ldots & & 7 \\ {\Sigma_3}^4(w) = 212210\ldots & & 5 \\ {\Sigma_3}^5(w) = 12210\ldots & & 3 \\ {\Sigma_3}^6(w) = 2210\ldots & & 6 \end{array}\right\}$$

lexicographic order of the shifted words

Some facts about shifts

Theorem (Amigó-E.-Kennel)

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Σ_N has exactly 6 forbidden patterns of length $N+2$.

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Example

The shortest forbidden patterns of Σ_4 are

$$615243, 324156, 342516, 162534, 453621, 435261.$$

The smallest # of letters needed to realize a pattern

For $\pi \in \mathcal{S}_n$, let

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- Let $\hat{\pi}$ be the cycle $(\pi_1, \pi_2, \dots, \pi_n)$ with the entry π_1 replaced with a $\star$.
- Let $\operatorname{des}(\hat{\pi})$ be the number of descents in $\hat{\pi}$ skipping the $\star$.

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Example

$$\pi = 892364157 \rightsquigarrow (8,9,2,3,6,4,1,5,7) \rightsquigarrow 536174892 \rightsquigarrow 536174{\star}92 = \hat{\pi}$$

$$\mathrm{des}(536174{\star}92) = 4$$

Theorem (E.)

$$N(\pi) = 1 + \mathrm{des}(\hat{\pi}) + \epsilon(\hat{\pi}),$$

where

$$\epsilon(\hat{\pi}) = \begin{cases} 1 & \text{if } \hat{\pi} = \star 1 \dots \text{ or } \hat{\pi} = \dots n\star, \\ 0 & \text{otherwise.} \end{cases}$$

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Example

$$N(892364157) = 1 + 4 + 0 = 5,$$

$$N(1423) = N(2134) = N(2314) = N(3241) = N(3421) = N(4132) = 3,$$

$$N(\pi) = 2 \text{ for all other } \pi \in \mathcal{S}_4.$$

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The distribution of the statistic $N(\pi)$ is related to the distribution of the number of descents in cyclic permutations.

Descent sets of 5-cycles

$\mathcal{C}_5$	
$(1,2,3,4,5) = 2345{\cdot}1$	
$(2,1,3,4,5) = 3{\cdot}145{\cdot}2$	
$(3,2,1,4,5) = 4{\cdot}125{\cdot}3$	
$(4,3,2,1,5) = 5{\cdot}1234$	
$(1,3,2,4,5) = 34{\cdot}25{\cdot}1$	
$(1,4,3,2,5) = 45{\cdot}23{\cdot}1$	
$(3,1,2,4,5) = 24{\cdot}15{\cdot}3$	
$(3,1,4,2,5) = 45{\cdot}123$	
$(4,3,1,2,5) = 25{\cdot}134$	
$(1,2,4,3,5) = 245{\cdot}3{\cdot}1$	
$(2,4,1,3,5) = 345{\cdot}12$	
$(4,1,2,3,5) = 235{\cdot}14$	

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$(2,3,1,4,5) = 4{\cdot}3{\cdot}15{\cdot}2$	
$(2,4,3,1,5) = 5{\cdot}4{\cdot}13{\cdot}2$	
$(4,2,3,1,5) = 5{\cdot}3{\cdot}124$	
$(1,4,2,3,5) = 4{\cdot}35{\cdot}2{\cdot}1$	
$(2,1,4,3,5) = 4{\cdot}15{\cdot}3{\cdot}2$	
$(2,3,4,1,5) = 5{\cdot}34{\cdot}12$	
$(3,4,2,1,5) = 5{\cdot}14{\cdot}23$	
$(4,2,1,3,5) = 3{\cdot}15{\cdot}24$	
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Main theorem

Theorem

For every n there is a bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$ *such that if* $\pi \in \mathcal{C}_{n+1}$ *and* $\sigma = \varphi(\pi)$, *then*

$$D(\pi) \cap [n-1] = D(\sigma).$$

The bijection $\varphi : \mathcal{C}_{n+1} \longrightarrow \mathcal{S}_n$; first step

Given $\pi \in \mathcal{C}_{n+1}$, write it in cycle form with $n+1$ at the end:

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21) \in \mathcal{C}_{21}$$

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Delete $n+1$ and split at the "left-to-right maxima":

$\sigma = (\underline{11}, 4, 10, 1, 7)(\underline{16}, 9, 3, 5, 12)(\underline{20}, 2, 6, 14, 18, 8, 13, 19, 15, 17) \in \mathcal{S}_{20}$.

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This map $\pi \mapsto \sigma$ is a bijection, but unfortunately it does not always preserve the descent set:

$$\pi(7) = 16 > \pi(8) = 13 \quad \text{but} \quad \sigma(7) = 11 < \sigma(8) = 13.$$

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We say that the pair $\{7, 8\}$ is *bad*. We will fix the bad pairs.

The bijection $\varphi : \mathcal{C}_{n+1} \longrightarrow \mathcal{S}_n$; fixing bad pairs

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
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For each but the last cycle of σ, from left to right:

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$\{7, 6\}$ and $\{7, 8\}$ are bad; and $\sigma(6) = 14 > 13 = \sigma(8) \Rightarrow \varepsilon := -1$.

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- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).

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- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$

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Switch 7 and 6. Switch 1 and 2.

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For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 10, 2, 6)(16, 9, 3, 5, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 7$. $\varepsilon := -1$.

Switch 7 and 6. Switch 1 and 2.

The bijection $\varphi : \mathcal{C}_{n+1} \longrightarrow \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, \underline{10}, 2, 6)(16, 9, 3, 5, 12)(\underline{20}, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 7$. $\varepsilon := -1$.

Switch 7 and 6. Switch 1 and 2.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 10, 2, 6)(16, 9, 3, 5, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 6.$ $\varepsilon := -1.$

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 10, 2, 6)(16, 9, 3, 5, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 6$. $\{6, 5\}$ is bad. $\varepsilon := -1$.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 10, 2, 6)(16, 9, 3, 5, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 6$. $\qquad \varepsilon := -1$.

Switch 6 and 5.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- z := rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. z := new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 10, 2, 5)(16, 9, 3, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 6$. $\varepsilon := -1$.

Switch 6 and 5.

The bijection $\varphi : \mathcal{C}_{n+1} \longrightarrow \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 10, 2, 5)(16, 9, 3, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 6$. $\varepsilon := -1$.

Switch 6 and 5. Switch 2 and 3.

The bijection $\varphi : \mathcal{C}_{n+1} \longrightarrow \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 10, 3, 5)(16, 9, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 6$. $\varepsilon := -1$.

Switch 6 and 5. Switch 2 and 3.

The bijection $\varphi : \mathcal{C}_{n+1} \longrightarrow \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 10, 3, 5)(16, 9, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 6$. $\varepsilon := -1$.

Switch 6 and 5. Switch 2 and 3. Switch 10 and 9.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, \underline{4}, 9, 3, 5)(\underline{16}, 10, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 6.$ $\varepsilon := -1.$

Switch 6 and 5. Switch 2 and 3. Switch 10 and 9.

The bijection $\varphi : \mathcal{C}_{n+1} \longrightarrow \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 5$. $\varepsilon := -1$.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 5$. $\{5, 4\}$ is OK, so we move on to the second cycle.

The bijection $\varphi : \mathcal{C}_{n+1} \longrightarrow \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 12$.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 12$. $\{12, 11\}$ is OK but $\{12, 13\}$ is bad $\Rightarrow \varepsilon := 1$.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 12$. $\varepsilon := 1$.

Switch 12 and 13.

The bijection $\varphi : \mathcal{C}_{n+1} \longrightarrow \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 13)(20, 1, 7, 14, 18, 8, 12, 19, 15, 17)$$

$z := 12$. $\varepsilon := 1$.

Switch 12 and 13.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, \underline{6}, 13)(20, 1, 7, 14, 18, \underline{8}, 12, 19, 15, 17)$$

$z := 12$. $\varepsilon := 1$.

Switch 12 and 13.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 13)(20, 1, 7, 14, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\varepsilon := 1$.

The bijection $\varphi : \mathcal{C}_{n+1} \rightarrow \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 13)(20, 1, 7, 14, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\{13, 14\}$ is bad. $\varepsilon := 1$.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 13)(20, 1, 7, 14, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\varepsilon := 1$.

Switch 13 and 14.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 14)(20, 1, 7, 13, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\varepsilon := 1$.

Switch 13 and 14.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 14)(20, 1, 7, 13, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\varepsilon := 1$.

Switch 13 and 14. Switch 6 and 7.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 7, 14)(20, 1, 6, 13, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\varepsilon := 1$.

Switch 13 and 14. Switch 6 and 7.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 7, 14)(20, 1, 6, 13, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\varepsilon := 1$.

Switch 13 and 14. Switch 6 and 7. Switch 2 and 1.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, \underline{10}, 1, 7, 14)(\underline{20}, 2, 6, 13, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\varepsilon := 1$.

Switch 13 and 14. Switch 6 and 7. Switch 2 and 1.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 14)(20, 2, 6, 13, 18, 8, 12, 19, 15, 17)$$

$z := 14$. $\varepsilon := 1$.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$; fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 14)(20, 2, 6, 13, 18, 8, 12, 19, 15, 17)$$

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The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$; fixing bad pairs

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Define $\varphi(\pi) = \sigma$.

The descent sets are preserved

$$\begin{aligned}\pi &= (11,4,10,1,7,16,9,3,5,12,20,2,6,14,18,8,13,19,15,17,21)\\ \varphi(\pi) &= (11,4,9,3,5)(16,10,1,7,15)(20,2,6,13,18,8,12,19,14,17)\end{aligned}$$

In one-line notation,

$$\begin{aligned}\pi &= 7\cdot 6\cdot 5\ 10\ 12\ 14\ 16\cdot 13\cdot 3\cdot 1\ 4\ 20\cdot 19\cdot 18\cdot 16\cdot\ 9\ \ 21\cdot 8\ 15\cdot 2\ 11\\ \varphi(\pi) &= 7\cdot 6\cdot 5\ \ 9\ \ 11\ 13\ 15\cdot 12\cdot 3\cdot 1\ 4\ 19\cdot 18\cdot 17\cdot 16\cdot 10\ 20\cdot 8\ 14\cdot 2\end{aligned}$$

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In fact, the set of weak excedances is preserved by φ as well.

The inverse map $\varphi^{-1} : \mathcal{S}_n \longrightarrow \mathcal{C}_{n+1}$

Given $\sigma \in \mathcal{S}_n$, write it in cycle form with the largest element of each cycle first, ordering the cycles by increasing first element:

$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17) \in \mathcal{S}_{20}.$$

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A pair $\{i, i + 1\}$ is *bad* if $\pi(i) > \pi(i + 1)$ but $\sigma(i) < \sigma(i + 1)$, or viceversa.

To find $\varphi^{-1}(\pi)$, we fix bad pairs in a similar way as before, now going from right to left. This undoes the switches performed by φ.

Necklaces

$A = \{x_1, x_2, \dots\}_<$ linearly ordered alphabet.

A *necklace* of length ℓ is a circular arrangement of ℓ beads labeled with elements of A, up to cyclic rotation.

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Given a multiset of necklaces,

- its *cycle structure* is the partition whose parts are the lengths of the necklaces;
- its *evaluation* is the monomial $x_1^{e_1} x_2^{e_2} \dots$ where e_i is the number of beads with label x_i.

Permutations and necklaces

Theorem (Gessel, Reutenauer '93)

Let $I = \{i_1, i_2, \ldots, i_k\}_< \subseteq [n-1]$, $\lambda \vdash n$. *Then*

$$|\{\pi \in \mathcal{S}_n \text{ with cycle structure } \lambda \text{ and } D(\pi) \subseteq I\}| = |\{\text{multisets of necklaces with cycle structure } \lambda \text{ and evaluation } x_1^{i_1} x_2^{i_2-i_1} \ldots x_k^{i_k-i_{k-1}} x_{k+1}^{n-i_k}\}|.$$

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This can be used to obtain a non-bijective proof of our result

$$|\{\pi \in \mathcal{C}_{n+1} : D(\pi) \cap [n-1] = I\}| = |\{\sigma \in \mathcal{S}_n : D(\sigma) = I\}|.$$

Non-bijective proof

Goal : $|\{\pi \in \mathcal{C}_{n+1} \,:\, D(\pi) \cap [n-1] = I\}| = |\{\sigma \in \mathcal{S}_n \,:\, D(\sigma) = I\}|.$

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Let $I = \{i_1, i_2, \dots, i_k\}_<$, $I' = I \cup \{n\}$. By the previous theorem,

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for all $I \subseteq [n-1]$. Now apply inclusion-exclusion.

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Let $\mathcal{T}_n$ be the set of n-cycles in one-line notation in which one entry has been replaced with 0.

$$\mathcal{T}_3 = \{031, 201, 230, 012, 302, 310\}.$$

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Corollary

For every n there is a bijection between $\mathcal{T}_n$ and $\mathcal{S}_n$ preserving the descent set.

Example:

$\mathcal{S}_3$	123	13·2	2·13	23·1	3·12	3·2·1
$\mathcal{T}_3$	012	03·1	3·02	23·0	3·02	3·1·0

A bijection preserving the cycle structure

Problem (Eriksen, Freij, Wästlund)

For any $I, J \subseteq [n-1]$ *with the same associated partition, give a bijection between derangements of* $[n]$ *whose descent set is contained in* I *and derangements of* $[n]$ *whose descent set is contained in* J.

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We can solve a generalization of this problem using the work of Gessel and Reutenauer:

Proposition

For any $I, J \subseteq [n-1]$ with the same associated partition, there exists a bijection $\{\pi \in \mathcal{S}_n : D(\pi) \subseteq I\} \to \{\sigma \in \mathcal{S}_n : D(\sigma) \subseteq J\}$ preserving the cycle structure.

THANK YOU