Permutations and β-shifts

Sergi Elizalde

Dartmouth College

October 6, 2010

Deterministic or random?

Two sequences of numbers in $[0, 1]$:

.6416, .9198, .2951, .8320, .5590, .9861, .0550, .2078, .6584, .8996, .3612, .9230, .2844, .8141, .6054, . . .

.9129, .5257, .4475, .9815, .4134, .9930, .1576, .8825, .3391, .0659, .1195, .5742, .1507, .5534, .0828, . . .

Which one is random? Which one is deterministic?

Deterministic or random?

Two sequences of numbers in $[0,1]$:

$.6416,\ .9198,\ .2951,\ .8320,\ .5590,\ .9861,\ .0550,\ .2078,\ .6584,\ .8996,\ .3612,\ .9230,\ .2844,\ .8141,\ .6054,\dots$

$.9129,\ .5257,\ .4475,\ .9815,\ .4134,\ .9930,\ .1576,\ .8825,\ .3391,\ .0659,\ .1195,\ .5742,\ .1507,\ .5534,\ .0828,\dots$

Which one is random? Which one is deterministic?

The first one is deterministic: taking $f(x) = 4x(1-x)$, we have

$f(.6146) = .9198,$
$f(.9198) = .2951,$
$f(.2951) = .8320,$
$\dots$

Allowed patterns of a map

Let X be a linearly ordered set, $f : X \to X$. For each $x \in X$ and $n \geq 1$, consider the sequence

$$x, f(x), f^2(x), \dots, f^{n-1}(x).$$

Allowed patterns of a map

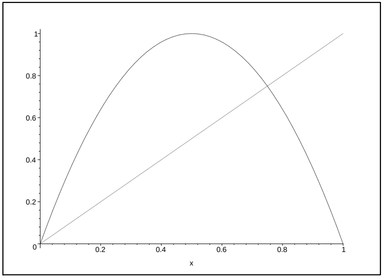
Let X be a linearly ordered set, $f : X \to X$. For each $x \in X$ and $n \geq 1$, consider the sequence

$$x, f(x), f^2(x), \dots, f^{n-1}(x).$$

If there are no repetitions, the relative order of the entries determines a permutation, called an allowed pattern of f.

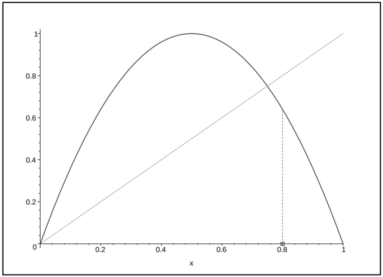
Example

$$\begin{aligned} f : \quad [0,1] &\longrightarrow [0,1] \\ x &\longmapsto 4x(1-x). \end{aligned}$$



Example

$$\begin{array}{rccl} f: & [0,1] & \longrightarrow & [0,1] \\ & x & \longmapsto & 4x(1-x). \end{array}$$

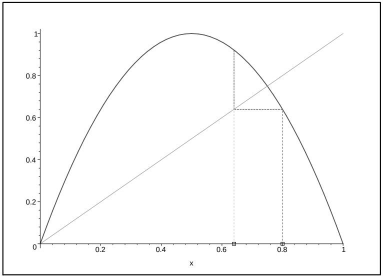


For $x = 0.8$ and $n = 4$, the sequence

$$0.8,$$

Example

$$\begin{array}{rcl} f: \quad [0,1] & \longrightarrow & [0,1] \\ x & \longmapsto & 4x(1-x). \end{array}$$

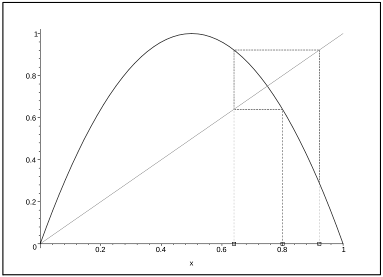


For $x = 0.8$ and $n = 4$, the sequence

$$0.8, \;\; 0.64,$$

Example

$$\begin{array}{rccc} f: & [0,1] & \longrightarrow & [0,1] \\ & x & \longmapsto & 4x(1-x). \end{array}$$

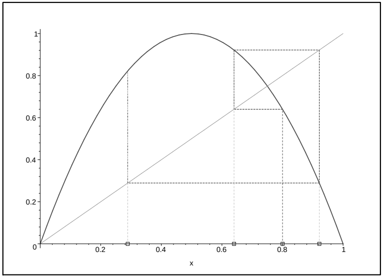


For $x = 0.8$ and $n = 4$, the sequence

$$0.8, \;\; 0.64, \;\; 0.9216,$$

Example

$$\begin{aligned} f : \quad [0,1] &\longrightarrow [0,1] \\ x &\longmapsto 4x(1-x). \end{aligned}$$

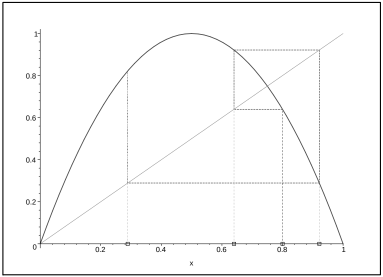


For $x = 0.8$ and $n = 4$, the sequence

$$0.8,\ 0.64,\ 0.9216,\ 0.2890$$

Example

$$\begin{array}{rccl} f: & [0,1] & \longrightarrow & [0,1] \\ & x & \longmapsto & 4x(1-x). \end{array}$$



For $x = 0.8$ and $n = 4$, the sequence

$$0.8,\ 0.64,\ 0.9216,\ 0.2890$$

determines the permutation 3241, so it is an allowed pattern.

Allowed and forbidden patterns

$\text{Allow}(f) =$ set of allowed patterns of f.

Allowed and forbidden patterns

$$\text{Allow}(f) = \text{set of allowed patterns of } f.$$

Allow(f) is closed under consecutive pattern containment.

E.g., if $4156273 \in \text{Allow}(f)$, then $2314 \in \text{Allow}(f)$.

Allowed and forbidden patterns

$$\text{Allow}(f) = \text{set of allowed patterns of } f.$$

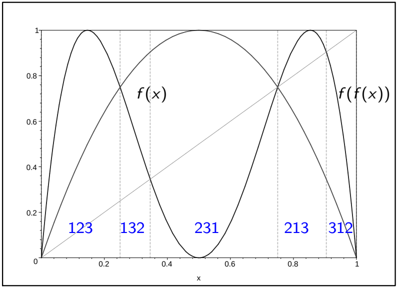
Allow(f) is closed under consecutive pattern containment.

E.g., if $4156273 \in \text{Allow}(f)$, then $2314 \in \text{Allow}(f)$.

The other permutations are called forbidden patterns of f.

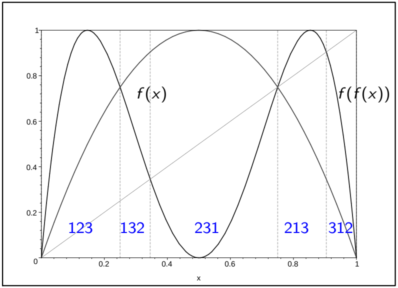
Example: $f(x) = 4x(1-x)$

Taking different $x \in [0,1]$, the patterns $123, 132, 231, 213, 312$ are realized. However, 321 is a forbidden pattern of f.



Example: $f(x) = 4x(1-x)$

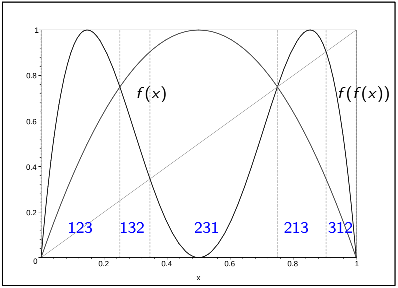
Taking different $x \in [0,1]$, the patterns $123, 132, 231, 213, 312$ are realized. However, 321 is a forbidden pattern of f.



Also forbidden: $\underbrace{1\mathbf{432}, 2\mathbf{431}, \mathbf{321}4, \ldots}_{\text{anything containing } 321}$

Example: $f(x) = 4x(1-x)$

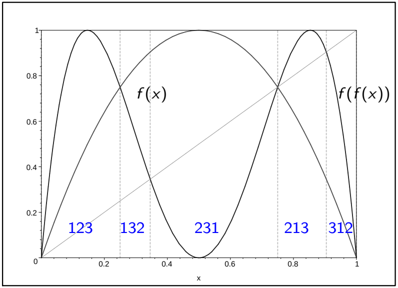
Taking different $x \in [0,1]$, the patterns 123, 132, 231, 213, 312 are realized. However, 321 is a forbidden pattern of f.



Also forbidden: $\underbrace{1\mathbf{432}, 2\mathbf{431}, \mathbf{321}4, \dots}_{\text{anything containing } 321}, \underbrace{1423, 2134, 2143, 3142, 4231, \dots}_{\textit{basic}\text{: not containing smaller forbidden patterns}}$

Example: $f(x) = 4x(1-x)$

Taking different $x \in [0,1]$, the patterns $123, 132, 231, 213, 312$ are realized. However, 321 is a forbidden pattern of f.



Also forbidden: $\underbrace{1\mathbf{432}, 2\mathbf{431}, \mathbf{321}4, \dots,}_{\text{anything containing } 321}$ $\underbrace{1423, 2134, 2143, 3142, 4231, \dots}_{\textit{basic}\text{: not containing smaller forbidden patterns}}$

Theorem (E.-Liu): f has infinitely many basic forbidden patterns.

Forbidden patterns

Let $I \subset \mathbb{R}$ be a closed interval.

Theorem (Bandt-Keller-Pompe '02)

Every piecewise monotone map $f : I \longrightarrow I$ *has forbidden patterns.*

Forbidden patterns

Let $I \subset \mathbb{R}$ be a closed interval.

Theorem (Bandt-Keller-Pompe '02)

Every piecewise monotone map $f : I \longrightarrow I$ *has forbidden patterns.*

Piecewise monotone: there is a finite partition of I into intervals such that f is continuous and strictly monotone on each interval.

Forbidden patterns

Let $I \subset \mathbb{R}$ be a closed interval.

Theorem (Bandt-Keller-Pompe '02)

Every piecewise monotone map $f : I \longrightarrow I$ has forbidden patterns.

Piecewise monotone: there is a finite partition of I into intervals such that f is continuous and strictly monotone on each interval.

Understanding the set of forbidden patterns of a given f is a difficult problem in general.

Deterministic vs. random sequences

Back to the original sequence:

$$.6416,\ .9198,\ .2951,\ .8320,\ .5590,\ .9861,\ .0550,\ .2078,\ .6584,\ .8996,\ .3612,\ .9230,\ .2844,\ .8141,\ .6054, \dots$$

We see that the pattern 321 is missing from it.

Deterministic vs. random sequences

Back to the original sequence:

.6416, .9198, .2951, .8320, .5590, .9861, .0550, .2078, .6584, .8996, .3612, .9230, .2844, .8141, .6054, . . .

We see that the pattern 321 is missing from it.

This suggests that the sequence is of the form $x_{i+1} = f(x_i)$ for some f.

Deterministic vs. random sequences

Back to the original sequence:

.6416, .9198, .2951, .8320, .5590, .9861, .0550, .2078, .6584, .8996, .3612, .9230, .2844, .8141, .6054, . . .

We see that the pattern 321 is missing from it.

This suggests that the sequence is of the form $x_{i+1} = f(x_i)$ for some f.

If it was a random sequence, any pattern would eventually appear.

Shift maps

For $N \geq 2$, let $\mathcal{W}_N = \{0, 1, \ldots, N-1\}^{\mathbb{N}}$ be the set of infinite words on N letters, equipped with the lexicographic order.

Shift maps

For $N \geq 2$, let $\mathcal{W}_N = \{0, 1, \ldots, N-1\}^{\mathbb{N}}$ be the set of infinite words on N letters, equipped with the lexicographic order.

Define the *shift* on N letters:

$$\begin{array}{cccc} \Sigma_N : & \mathcal{W}_N & \longrightarrow & \mathcal{W}_N \\ & w_1 w_2 w_3 \ldots & \mapsto & w_2 w_3 w_4 \ldots \end{array}$$

Shift maps

For $N \geq 2$, let $\mathcal{W}_N = \{0, 1, \ldots, N{-}1\}^{\mathbb{N}}$ be the set of infinite words on N letters, equipped with the lexicographic order.

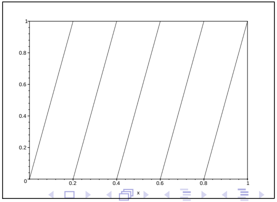
Define the *shift* on N letters:

$$\begin{array}{cccc} \Sigma_N : & \mathcal{W}_N & \longrightarrow & \mathcal{W}_N \\ & w_1 w_2 w_3 \ldots & \mapsto & w_2 w_3 w_4 \ldots \end{array}$$

Thinking of words as expansions in base N of numbers in $[0, 1)$, Σ_N is "equivalent" to the *sawtooth map*

$$\begin{array}{cccc} M_N : & [0, 1) & \rightarrow & [0, 1) \\ & x & \mapsto & \{Nx\} \end{array}$$

(fractional part)



Example

The permutation 4217536 is realized (i.e., allowed) by Σ_3, because taking $w = 2102212210\ldots \in \mathcal{W}_3$, we have

$$\left.\begin{array}{rcll} w &=& 2102212210\ldots & 4 \\ \Sigma_3(w) &=& 102212210\ldots & 2 \\ {\Sigma_3}^2(w) &=& 02212210\ldots & 1 \\ {\Sigma_3}^3(w) &=& 2212210\ldots & 7 \\ {\Sigma_3}^4(w) &=& 212210\ldots & 5 \\ {\Sigma_3}^5(w) &=& 12210\ldots & 3 \\ {\Sigma_3}^6(w) &=& 2210\ldots & 6 \end{array}\right\} \begin{array}{l} \text{lexicographic order} \\ \text{of the shifted words} \end{array}$$

Example

The permutation 4217536 is realized (i.e., allowed) by Σ_3, because taking $w = 2102212210\ldots \in \mathcal{W}_3$, we have

$$\left.\begin{array}{rl} w = 2102212210\ldots & 4 \\ \Sigma_3(w) = 102212210\ldots & 2 \\ {\Sigma_3}^2(w) = 02212210\ldots & 1 \\ {\Sigma_3}^3(w) = 2212210\ldots & 7 \\ {\Sigma_3}^4(w) = 212210\ldots & 5 \\ {\Sigma_3}^5(w) = 12210\ldots & 3 \\ {\Sigma_3}^6(w) = 2210\ldots & 6 \end{array}\right\} \begin{array}{l}\text{lexicographic order}\\ \text{of the shifted words}\end{array}$$

We say that w *induces* 4217536.

Forbidden patterns of shifts

Theorem (Amigó-E.-Kennel)

Σ_N *has no forbidden patterns of length* $n \leq N+1$*, but it has forbidden patterns of each length* $n \geq N+2$*.*

Forbidden patterns of shifts

Theorem (Amigó-E.-Kennel)

Σ_N *has no forbidden patterns of length* $n \leq N+1$*, but it has forbidden patterns of each length* $n \geq N+2$.

Proposition (E.)

Σ_N *has exactly 6 forbidden patterns of length* $N+2$.

Forbidden patterns of shifts

Theorem (Amigó-E.-Kennel)

Σ_N *has no forbidden patterns of length* $n \leq N+1$, *but it has forbidden patterns of each length* $n \geq N+2$.

Proposition (E.)

Σ_N *has exactly 6 forbidden patterns of length* $N+2$.

Example

The shortest forbidden patterns of Σ_4 are

$$615243, 324156, 342516, 162534, 453621, 435261.$$

The smallest # of letters needed to realize a pattern

For $\pi \in \mathcal{S}_n$, let $\quad N(\pi) = \min\{N : \pi \in \mathrm{Allow}(\Sigma_N)\}$.

The smallest # of letters needed to realize a pattern

For $\pi \in \mathcal{S}_n$, let $N(\pi) = \min\{N : \pi \in \text{Allow}(\Sigma_N)\}$.

Theorem (E.): $N(\pi) = 1 + \text{des}(\hat{\pi}) + \underbrace{\epsilon(\hat{\pi})}_{0 \text{ or } 1}$.

The smallest # of letters needed to realize a pattern

For $\pi \in \mathcal{S}_n$, let $\quad N(\pi) = \min\{N : \pi \in \text{Allow}(\Sigma_N)\}$.

Theorem (E.): $\quad N(\pi) = 1 + \text{des}(\hat{\pi}) + \underbrace{\epsilon(\hat{\pi})}_{0 \text{ or } 1}$.

An example of what $\hat{\pi}$ is:

$\pi = 892364157 \rightsquigarrow (8,9,2,3,6,4,1,5,7) \rightsquigarrow 536174892 \rightsquigarrow 536174{\star}92 = \hat{\pi}$

The smallest # of letters needed to realize a pattern

For $\pi \in \mathcal{S}_n$, let $N(\pi) = \min\{N : \pi \in \text{Allow}(\Sigma_N)\}$.

Theorem (E.): $N(\pi) = 1 + \text{des}(\hat{\pi}) + \underbrace{\epsilon(\hat{\pi})}_{0 \text{ or } 1}$.

An example of what $\hat{\pi}$ is:

$$\pi = 892364157 \rightsquigarrow (8,9,2,3,6,4,1,5,7) \rightsquigarrow 536174892 \rightsquigarrow 536174{\star}92 = \hat{\pi}$$

$$\text{des}(\hat{\pi}) = \text{des}(536174{\star}92) = 4$$

The smallest # of letters needed to realize a pattern

For $\pi \in \mathcal{S}_n$, let $\quad N(\pi) = \min\{N : \pi \in \mathrm{Allow}(\Sigma_N)\}$.

Theorem (E.): $\quad N(\pi) = 1 + \mathrm{des}(\hat{\pi}) + \underbrace{\epsilon(\hat{\pi})}_{0 \text{ or } 1}$.

An example of what $\hat{\pi}$ is:

$$\pi = 892364157 \rightsquigarrow (8,9,2,3,6,4,1,5,7) \rightsquigarrow 536174892 \rightsquigarrow 536174{\star}92 = \hat{\pi}$$

$$\mathrm{des}(\hat{\pi}) = \mathrm{des}(536174{\star}92) = 4$$

$$N(892364157) = 1 + 4 + 0 = 5$$

The smallest # of letters needed to realize a pattern

For $\pi \in \mathcal{S}_n$, let $N(\pi) = \min\{N : \pi \in \text{Allow}(\Sigma_N)\}$.

Theorem (E.): $N(\pi) = 1 + \text{des}(\hat{\pi}) + \underbrace{\epsilon(\hat{\pi})}_{0 \text{ or } 1}.$

An example of what $\hat{\pi}$ is:

$$\pi = 892364157 \rightsquigarrow (8,9,2,3,6,4,1,5,7) \rightsquigarrow 536174892 \rightsquigarrow 536174{\star}92 = \hat{\pi}$$

$$\text{des}(\hat{\pi}) = \text{des}(536174{\star}92) = 4$$

$$N(892364157) = 1 + 4 + 0 = 5$$

We have a (complicated) formula for the number of permutations in $\mathcal{S}_n$ that are realized by Σ_N, for given n and N.

β-shifts

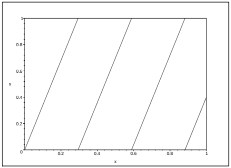
- Natural generalization of shifts.
- Widely studied in the literature from the perspective of measure theory, automata theory, and number theory.

β-shifts

- Natural generalization of shifts.
- Widely studied in the literature from the perspective of measure theory, automata theory, and number theory.

For a real number $\beta > 1$, let M_β be the β-sawtooth map

$$\begin{array}{rccc} M_\beta : & [0,1) & \longrightarrow & [0,1) \\ & x & \longmapsto & \{\beta x\} \end{array}$$

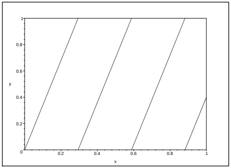


β-shifts

- Natural generalization of shifts.
- Widely studied in the literature from the perspective of measure theory, automata theory, and number theory.

For a real number $\beta > 1$, let M_β be the β-sawtooth map

$$\begin{array}{rccc} M_\beta : & [0,1) & \to & [0,1) \\ & x & \mapsto & \{\beta x\} \end{array}$$



We would like to define the β-shift as

$$\begin{array}{rccc} \Sigma_\beta : & W(\beta) & \longrightarrow & W(\beta) \\ & w_1 w_2 w_3 \ldots & \mapsto & w_2 w_3 w_4 \ldots \end{array}$$

for some set $W(\beta)$.

The domain of Σ_β

$$\begin{array}{cccc} \Sigma_\beta : & W(\beta) & \longrightarrow & W(\beta) \\ & w_1 w_2 w_3 \ldots & \mapsto & w_2 w_3 w_4 \ldots \end{array}$$

The domain of Σ_β

$$\begin{array}{rccc} \Sigma_\beta : & W(\beta) & \longrightarrow & W(\beta) \\ & w_1 w_2 w_3 \ldots & \mapsto & w_2 w_3 w_4 \ldots \end{array}$$

For M_β and Σ_β to be "equivalent", $W(\beta)$ should be the set of words given by expansions in base β of numbers $x \in [0, 1)$:

The domain of Σ_β

$$\begin{array}{rccc} \Sigma_\beta : & W(\beta) & \longrightarrow & W(\beta) \\ & w_1 w_2 w_3 \dots & \mapsto & w_2 w_3 w_4 \dots \end{array}$$

For M_β and Σ_β to be "equivalent", $W(\beta)$ should be the set of words given by expansions in base β of numbers $x \in [0,1)$:

$$x = \frac{w_1}{\beta} + \frac{w_2}{\beta^2} + \cdots ,$$

with

$$\begin{aligned} w_1 &= \lfloor \beta x \rfloor, \\ w_2 &= \lfloor \beta \{\beta x\} \rfloor, \\ w_3 &= \lfloor \beta \{\beta \{\beta x\}\} \rfloor, \\ \dots \end{aligned}$$

The domain of Σ_β

$$\begin{aligned} \Sigma_\beta : \quad W(\beta) & \longrightarrow W(\beta) \\ w_1 w_2 w_3 \ldots & \mapsto w_2 w_3 w_4 \ldots \end{aligned}$$

Theorem (Parry '60)

Let

$$\beta = a_0 + \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots$$

be the β-expansion of β. Then (up to small technicalities)

$$W(\beta) = \{w_1 w_2 w_3 \ldots : w_k w_{k+1} w_{k+2} \ldots <_{lex} a_0 a_1 a_2 \ldots \text{ for all } k \geq 1\}.$$

The domain of Σ_β

$$\begin{array}{rccc} \Sigma_\beta : & W(\beta) & \longrightarrow & W(\beta) \\ & w_1 w_2 w_3 \ldots & \mapsto & w_2 w_3 w_4 \ldots \end{array}$$

$$W(\beta) = \{w_1 w_2 w_3 \ldots : w_k w_{k+1} w_{k+2} \ldots <_{lex} a_0 a_1 a_2 \ldots \text{ for all } k \geq 1\}.$$

The domain of Σ_β

$$\begin{array}{cccc} \Sigma_\beta : & W(\beta) & \longrightarrow & W(\beta) \\ & w_1 w_2 w_3 \ldots & \mapsto & w_2 w_3 w_4 \ldots \end{array}$$

$$W(\beta) = \{w_1 w_2 w_3 \ldots : w_k w_{k+1} w_{k+2} \ldots <_{lex} a_0 a_1 a_2 \ldots \text{ for all } k \geq 1\}.$$

Example

- For $\beta = N \in \mathbb{Z}$, $\qquad a_0 a_1 a_2 \ldots = N00 \ldots = N0^\infty$, $W(N) = \mathcal{W}_N = \{0, 1, \ldots, N-1\}^{\mathbb{N}}$.

The domain of Σ_β

$$\begin{array}{rccc} \Sigma_\beta : & W(\beta) & \longrightarrow & W(\beta) \\ & w_1 w_2 w_3 \ldots & \mapsto & w_2 w_3 w_4 \ldots \end{array}$$

$$W(\beta) = \{w_1 w_2 w_3 \ldots : w_k w_{k+1} w_{k+2} \ldots <_{lex} a_0 a_1 a_2 \ldots \text{ for all } k \geq 1\}.$$

Example

- For $\beta = N \in \mathbb{Z}$, $\quad a_0 a_1 a_2 \ldots = N00 \ldots = N0^\infty$, $W(N) = \mathcal{W}_N = \{0, 1, \ldots, N-1\}^{\mathbb{N}}$.
- For $\beta = 1 + \sqrt{2}$, $\quad a_0 a_1 a_2 \ldots = 210^\infty$, $W(\beta) =$ words over $\{0, 1, 2\}$ where every 2 is followed by a 0.

The shift-complexity of a permutation

It can be shown that if $1 < \beta \leq \beta'$, then

- $W(\beta) \subseteq W(\beta')$,
- $\mathrm{Allow}(\Sigma_\beta) \subseteq \mathrm{Allow}(\Sigma_{\beta'})$.

The shift-complexity of a permutation

It can be shown that if $1 < \beta \leq \beta'$, then

- $W(\beta) \subseteq W(\beta')$,
- $\mathrm{Allow}(\Sigma_\beta) \subseteq \mathrm{Allow}(\Sigma_{\beta'})$.

Definition (*shift-complexity*)

For any permutation π, let

$$B(\pi) = \inf\{\beta : \pi \in \mathrm{Allow}(\Sigma_\beta)\}.$$

The shift-complexity of a permutation

It can be shown that if $1 < \beta \leq \beta'$, then

- $W(\beta) \subseteq W(\beta')$,
- $\text{Allow}(\Sigma_\beta) \subseteq \text{Allow}(\Sigma_{\beta'})$.

Definition (*shift-complexity*)

For any permutation π, let

$$B(\pi) = \inf\{\beta : \pi \in \text{Allow}(\Sigma_\beta)\}.$$

We have that $N(\pi) = \lfloor B(\pi) \rfloor + 1$.

Our goal is to be able to determine $B(\pi)$ for an arbitrary π.

Another characterization of $B(\pi)$: statistics on words

For an infinite word $w = w_1 w_2 \ldots \neq 0^\infty$, let

- $\hat{b}(w) =$ unique solution with $\beta \geq 1$ of

$$\frac{w_1}{\beta} + \frac{w_2}{\beta^2} + \cdots + \frac{w_n}{\beta^n} + \cdots = 1,$$

Another characterization of $B(\pi)$: statistics on words

For an infinite word $w = w_1 w_2 \ldots \neq 0^\infty$, let

- $\hat{b}(w)$ = unique solution with $\beta \geq 1$ of

$$\frac{w_1}{\beta} + \frac{w_2}{\beta^2} + \cdots + \frac{w_n}{\beta^n} + \cdots = 1,$$

- $b(w) = \sup_{k\geq 1} \hat{b}(w_k w_{k+1} \ldots)$.

Another characterization of $B(\pi)$: statistics on words

For an infinite word $w = w_1 w_2 \ldots \neq 0^\infty$, let

- $\hat{b}(w) =$ unique solution with $\beta \geq 1$ of
$$\frac{w_1}{\beta} + \frac{w_2}{\beta^2} + \cdots + \frac{w_n}{\beta^n} + \cdots = 1,$$
- $b(w) = \sup_{k\geq 1} \hat{b}(w_k w_{k+1} \ldots)$.

Proposition (E.)

$$B(\pi) = \inf\{b(w) : w \text{ induces } \pi\}.$$

Another characterization of $B(\pi)$: statistics on words

For an infinite word $w = w_1 w_2 \ldots \neq 0^\infty$, let

- $\hat{b}(w) =$ unique solution with $\beta \geq 1$ of
$$\frac{w_1}{\beta} + \frac{w_2}{\beta^2} + \cdots + \frac{w_n}{\beta^n} + \cdots = 1,$$
- $b(w) = \sup_{k\geq 1} \hat{b}(w_k w_{k+1} \ldots)$.

Proposition (E.)

$$B(\pi) = \inf\{b(w) : w \text{ induces } \pi\}.$$

To compute $B(\pi)$, we'll find a word w inducing π such that $b(w)$ is as small as possible.

Example

Goal: find a word w inducing π such that $b(w)$ is small.

Example

Goal: find a word w inducing π such that $b(w)$ is small.

$$\begin{array}{rl} \pi = & 7\quad 3\quad 5\quad 4\quad 9\quad 1\quad 8\quad 2\quad 6 \\ w = & \end{array}$$

Example

Goal: find a word w inducing π such that $b(w)$ is small.

$$\begin{array}{rccccccccc} \pi = & 7 & 3 & 5 & 4 & 9 & 1 & 8 & 2 & 6 \\ w = & & & & & & 0 & & & \end{array}$$

Example

Goal: find a word w inducing π such that $b(w)$ is small.

$$\begin{array}{rccccccccc} \pi = & 7 & 3 & 5 & 4 & 9 & 1 & 8 & 2 & 6 \\ w = & & & & & & 0 & & 1 & \end{array}$$

Example

Goal: find a word w inducing π such that $b(w)$ is small.

$$\begin{array}{rccccccccc} \pi = & 7 & 3 & 5 & 4 & 9 & 1 & 8 & 2 & 6 \\ w = & & 2 & & & & 0 & & 1 & \end{array}$$

Example

Goal: find a word w inducing π such that $b(w)$ is small.

$$\begin{array}{rccccccccc} \pi = & 7 & 3 & 5 & 4 & 9 & 1 & 8 & 2 & 6 \\ w = & & 2 & & 2 & & 0 & & 1 & \end{array}$$

Example

Goal: find a word w inducing π such that $b(w)$ is small.

$$\begin{array}{rccccccccc} \pi = & 7 & 3 & 5 & 4 & 9 & 1 & 8 & 2 & 6 \\ w = & & 2 & 3 & 2 & & 0 & & 1 & \end{array}$$

Example

Goal: find a word w inducing π such that $b(w)$ is small.

$$\begin{array}{rccccccccc} \pi = & 7 & 3 & 5 & 4 & 9 & 1 & 8 & 2 & 6 \\ w = & & 2 & 3 & 2 & & 0 & & 1 & ? \end{array}$$

Example

Goal: find a word w inducing π such that $b(w)$ is small.

$$\begin{array}{rccccccccc} \pi = & 7 & 3 & 5 & 4 & 9 & 1 & 8 & 2 & 6 \\ w = & 4 & 2 & 3 & 2 & & 0 & & 1 & ? \end{array}$$

Example

Goal: find a word w inducing π such that $b(w)$ is small.

$$\begin{array}{rccccccccc} \pi = & 7 & 3 & 5 & 4 & 9 & 1 & 8 & 2 & 6 \\ w = & 4 & 2 & 3 & 2 & & 0 & 5 & 1 & ? \end{array}$$

Example

Goal: find a word w inducing π such that $b(w)$ is small.

$$\begin{array}{rccccccccc} \pi = & 7 & 3 & 5 & 4 & 9 & 1 & 8 & 2 & 6 \\ w = & 4 & 2 & 3 & 2 & 6 & 0 & 5 & 1 & ? \end{array}$$

Example

Goal: find a word w inducing π such that $b(w)$ is small.

$$\begin{array}{rccccccccc} \pi = & 7 & 3 & 5 & 4 & 9 & 1 & 8 & 2 & 6 \\ w = & 4 & 2 & 3 & 2 & 6 & 0 & 5 & 1 & ? \end{array}$$

Proposition

The entries $w_1 w_2 \ldots w_{n-1}$ are forced (if minimizing $\#$ letters)

Example

Goal: find a word w inducing π such that $b(w)$ is small.

$$\begin{array}{rccccccccc} \pi = & 7 & 3 & 5 & 4 & 9 & 1 & 8 & 2 & 6 \\ w = & 4 & 2 & 3 & 2 & 6 & 0 & 5 & 1 & ? \end{array}$$

Proposition

The entries $w_1 w_2 \dots w_{n-1}$ are forced (if minimizing $\#$ letters) , and they can be completed into a word w that induces π.

Example

Goal: find a word w inducing π such that $b(w)$ is small.

$$\begin{array}{rccccccccc} \pi = & 7 & 3 & 5 & 4 & 9 & 1 & 8 & 2 & 6 \\ w = & 4 & 2 & 3 & 2 & 6 & 0 & 5 & 1 & ? \end{array}$$

Proposition

The entries $w_1 w_2 \dots w_{n-1}$ are forced (if minimizing $\#$ letters) , and they can be completed into a word w that induces π.

Now we choose the entries $w_n w_{n+1} \dots$ in order to minimize $b(w)$.

Example

Goal: find a word w inducing π such that $b(w)$ is small.

$$\begin{array}{rcccccccccc} \pi = & 7 & 3 & 5 & 4 & 9 & 1 & 8 & 2 & 6 & \\ w = & 4 & 2 & 3 & 2 & 6 & 0 & 5 & 1 & 3 & 3 \; 0^\infty \end{array}$$

Proposition

The entries $w_1 w_2 \dots w_{n-1}$ are forced (if minimizing $\#$ letters) , and they can be completed into a word w that induces π.

Now we choose the entries $w_n w_{n+1} \dots$ in order to minimize $b(w)$.

Example

Goal: find a word w inducing π such that $b(w)$ is small.

$$\begin{array}{rccccccccccc} \pi = & 7 & 3 & 5 & 4 & 9 & 1 & 8 & 2 & 6 & & \\ w = & 4 & 2 & 3 & 2 & 6 & 0 & 5 & 1 & 3 & 3 & 0^{\infty} \end{array}$$

Proposition

The entries $w_1 w_2 \dots w_{n-1}$ are forced (if minimizing # letters) , and they can be completed into a word w that induces π.

Now we choose the entries $w_n w_{n+1} \dots$ in order to minimize $b(w)$.

In this example, $B(\pi) = b(42326051330^{\infty}) = \hat{b}(6051330^{\infty})$.

Another example

$$\pi = \; 8 \;\; 9 \;\; 3 \;\; 1 \;\; 4 \;\; 6 \;\; 2 \;\; 7 \;\; 5$$
$$w =$$

Another example

$$\begin{array}{rccccccccc}
\pi = & 8 & 9 & 3 & 1 & 4 & 6 & 2 & 7 & 5 \\
w = & & & & 0 & & & & &
\end{array}$$

Another example

$$\begin{array}{rccccccccc} \pi = & 8 & 9 & 3 & 1 & 4 & 6 & 2 & 7 & 5 \\ w = & & & & 0 & & & 0 & & \end{array}$$

Another example

$$\begin{array}{rccccccccc}
\pi = & 8 & 9 & 3 & 1 & 4 & 6 & 2 & 7 & 5 \\
w = & & & 1 & 0 & & & 0 & &
\end{array}$$

Another example

$$\begin{array}{rccccccccc}
\pi = & 8 & 9 & 3 & 1 & 4 & 6 & 2 & 7 & 5 \\
w = & & & 1 & 0 & 1 & & 0 & &
\end{array}$$

Another example

$$\begin{array}{rccccccccc}
\pi = & 8 & 9 & 3 & 1 & 4 & 6 & 2 & 7 & 5 \\
w = & & & 1 & 0 & 1 & & 0 & & ?
\end{array}$$

Another example

$$\begin{array}{rccccccccc}
\pi = & 8 & 9 & 3 & 1 & 4 & 6 & 2 & 7 & 5 \\
w = & & & 1 & 0 & 1 & 2 & 0 & & ?
\end{array}$$

Another example

$$\begin{array}{rccccccccc}
\pi = & 8 & 9 & 3 & 1 & 4 & 6 & 2 & 7 & 5 \\
w = & & & 1 & 0 & 1 & 2 & 0 & 2 & ?
\end{array}$$

Another example

$$\begin{array}{rccccccccc} \pi = & 8 & 9 & 3 & 1 & 4 & 6 & 2 & 7 & 5 \\ w = & 2 & & 1 & 0 & 1 & 2 & 0 & 2 & ? \end{array}$$

Another example

$$\begin{array}{rccccccccc} \pi = & 8 & 9 & 3 & 1 & 4 & 6 & 2 & 7 & 5 \\ w = & 2 & 3 & 1 & 0 & 1 & 2 & 0 & 2 & ? \end{array}$$

Another example

$$\begin{array}{llllllllllll} \pi = & 8 & 9 & 3 & 1 & 4 & 6 & 2 & 7 & 5 & & \\ w = & 2 & 3 & 1 & 0 & 1 & 2 & 0 & 2 & 1 & 3^\infty & \text{(one possibility)} \end{array}$$

Another example

$$\begin{array}{rcccccccccccccl}
\pi = & 8 & 9 & 3 & 1 & 4 & 6 & 2 & 7 & 5 & & & & & \\
w = & 2 & 3 & 1 & 0 & 1 & 2 & 0 & 2 & 1 & 3^{\infty} & & & & \text{(one possibility)} \\
w = & 2 & 3 & 1 & 0 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 2 & 0^{\infty} \quad \text{(smaller } b\text{)}
\end{array}$$

Another example

$$\begin{array}{rllllllllllllll} \pi = & 8 & 9 & 3 & 1 & 4 & 6 & 2 & 7 & 5 \\ w = & 2 & 3 & 1 & 0 & 1 & 2 & 0 & 2 & 1 & 3^{\infty} & \text{(one possibility)} \\ w = & 2 & 3 & 1 & 0 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 2 & 0^{\infty} & \text{(smaller } b\text{)} \end{array}$$

In this example, letting $w^{(m)} = 2310(1202)^m 20^{\infty}$, we have

$$B(\pi) = \lim_{m \to \infty} b(w^{(m)}).$$

Computation of $B(\pi)$ in general

Given a finite word $u_1 u_2 \dots u_r$, let

$$p_{u_1 u_2 \dots u_r}(\beta) = \beta^r - u_1\beta^{r-1} - u_2\beta^{r-2} - \dots - u_r.$$

Computation of $B(\pi)$ in general

Given a finite word $u_1 u_2 \dots u_r$, let

$$p_{u_1 u_2 \dots u_r}(\beta) = \beta^r - u_1 \beta^{r-1} - u_2 \beta^{r-2} - \dots - u_r.$$

Theorem (E.)

For $\pi \in \mathcal{S}_n$, let $c = \pi(n)$, $\ell = \pi^{-1}(n)$, $k = \pi^{-1}(c-1)$, and let $w_1 w_2 \dots w_{n-1}$ be the forced entries for w. Let

$$P_\pi(\beta) = \begin{cases} p_{w_\ell w_{\ell+1} \dots w_{n-1}}(\beta) & \text{if } c = 1, \\ p_{w_\ell w_{\ell+1} \dots w_{n-1} w_k w_{k+1} \dots w_{\ell-1}}(\beta) - 1 & \text{if } c \neq 1,\ \ell > k, \\ p_{w_\ell w_{\ell+1} \dots w_{n-1}}(\beta) - p_{w_\ell w_{\ell+1} \dots w_{k-1}}(\beta) & \text{if } c \neq 1,\ \ell < k. \end{cases}$$

Then $B(\pi)$ is the unique real root with $\beta \geq 1$ of $P_\pi(\beta)$.

Examples

- For $\pi = 735491826$, the forced entries are $w_1 \dots w_8 = 42326051$.

  Here, $B(735491826) \approx 6.139428921$ is the root with $\beta \geq 1$ of

$$P_\pi(\beta) = p_{605133}(\beta) - 1 = \beta^6 - 6\beta^5 - 5\beta^3 - \beta^2 - 3\beta - 3.$$

Examples

- For $\pi = 735491826$, the forced entries are $w_1 \dots w_8 = 42326051$.

  Here, $B(735491826) \approx 6.139428921$ is the root with $\beta \geq 1$ of
  $$P_\pi(\beta) = p_{605133}(\beta) - 1 = \beta^6 - 6\beta^5 - 5\beta^3 - \beta^2 - 3\beta - 3.$$

- For $\pi = 893146275$, the forced entries are $w_1 \dots w_8 = 23101202$.

  Here, $B(893146275) \approx 3.343618091$ is the root with $\beta \geq 1$ of
  $$P_\pi(\beta) = p_{3101202}(\beta) - p_{310}(\beta) = \beta^7 - 3\beta^6 - \beta^5 - 2\beta^3 + \beta^2 + \beta - 2.$$

The shift-complexity of permutations of length ≤ 4

$\pi \in \mathcal{S}_2$	$\pi \in \mathcal{S}_3$	$\pi \in \mathcal{S}_4$	$B(\pi)$	$B(\pi)$ is a root of
12, 21	123, 231, 312	1234, 2341, 3412, 4123	1	$\beta - 1$
		1342, 2413, 3124, 4231	1.465571232	$\beta^3 - \beta^2 - 1$
	132, 213, 321	1243, 1324, 2431, 3142, 4312	$\frac{1+\sqrt{5}}{2} \approx 1.618033989$	$\beta^2 - \beta - 1$
		4213	1.801937736	$\beta^3 - \beta^2 - 2\beta + 1$
		1432, 2143, 3214, 4321	1.839286755	$\beta^3 - \beta^2 - \beta - 1$
		2134, 3241	2	$\beta - 2$
		4132	2.246979604	$\beta^3 - 2\beta^2 - \beta + 1$
		2314, 3421	$1 + \sqrt{2} \approx 2.414213562$	$\beta^2 - 2\beta - 1$
		1423	$\frac{3+\sqrt{5}}{2} \approx 2.618033989$	$\beta^2 - 3\beta + 1$

The shift-complexity of permutations of length 5 (pg. 1)

$\pi \in \mathcal{S}_5$	$B(\pi)$	$B(\pi)$ is a root of
12345, 23451, 34512, 45123, 51234	1	$\beta - 1$
13452, 24513, 35124, 41235, 52341	1.380277569	$\beta^4 - \beta^3 - 1$
12453, 13524, 24135, 35241, 41352, 53412	1.465571232	$\beta^3 - \beta^2 - 1$
52413	1.558979878	$\beta^4 - \beta^3 - 2\beta + 1$
12354, 12435, 14253, 23541, 31425, 35412, 41253, 42531, 54123	$\frac{1+\sqrt{5}}{2} \approx 1.6180$	$\beta^2 - \beta - 1$
53124	1.722083806	$\beta^4 - \beta^3 - \beta^2 - \beta + 1$
13542, 25413, 31254, 43125, 54231	1.754877666	$\beta^3 - 2\beta^2 + \beta - 1$
25314, 53142	1.801937736	$\beta^3 - \beta^2 - 2\beta + 1$
12543, 13254, 14325, 25431, 31542, 42153, 54312	1.839286755	$\beta^3 - \beta^2 - \beta - 1$
54213	1.905166168	$\beta^4 - \beta^3 - 2\beta^2 + 1$
53214	1.921289610	$\beta^4 - \beta^3 - \beta^2 - 2\beta + 1$
15432, 21543, 32154, 43215, 54321	1.927561975	$\beta^4 - \beta^3 - \beta^2 - \beta - 1$
13245, 21345, 24351, 31245, 32145, 32451, 42351, 43251, 43512	2	$\beta - 2$
51342	2.117688633	$\beta^4 - 2\beta^3 - \beta + 1$
51243	$\frac{1+\sqrt{5+4\sqrt{2}}}{2} \approx 2.1322$	$\beta^4 - 2\beta^3 - \beta^2 + 2\beta - 1$
34125, 42513, 45231	2.205569430	$\beta^3 - 2\beta^2 - 1$
35142, 45132, 51324	2.246979604	$\beta^3 - 2\beta^2 - \beta + 1$
14352, 25143, 32514, 41325, 52431	2.277452390	$\beta^4 - 2\beta^3 - \beta - 1$
51432	2.296630263	$\beta^4 - 2\beta^3 - 2\beta + 1$
25134	2.324717957	$\beta^3 - 3\beta^2 + 2\beta - 1$

The shift-complexity of permutations of length 5 (pg. 2)

$\pi \in \mathcal{S}_5$	$B(\pi)$	$B(\pi)$ is a root of
23514, 31452	2.359304086	$\beta^3 - 2\beta^2 - 2$
13425, 23415, 24531, 34152, 34521, 43152, 45312	$1 + \sqrt{2} \approx 2.4142$	$\beta^2 - 2\beta - 1$
45213	2.481194304	$\beta^3 - 2\beta^2 - 2\beta + 2$
52143	2.496698205	$\beta^4 - 2\beta^3 - \beta^2 - \beta + 1$
52134	2.505068414	$\beta^4 - 3\beta^3 + \beta^2 + \beta - 1$
14532, 21453, 35214, 42135, 53241	2.521379707	$\beta^3 - 3\beta^2 + 2\beta - 2$
34215, 41532, 45321	2.546818277	$\beta^3 - 2\beta^2 - \beta - 1$
12534, 14523, 15234, 21534, 41523	$\frac{3+\sqrt{5}}{2} \approx 2.6180$	$\beta^2 - 3\beta + 1$
14235, 25341	2.658967082	$\beta^3 - 2\beta^2 - \beta - 2$
52314	2.691739510	$\beta^4 - 2\beta^3 - 2\beta^2 + 1$
15342, 24153, 31524, 42315, 53421	2.696797189	$\beta^4 - 2\beta^3 - \beta^2 - 2\beta - 1$
21354, 21435, 32541	$1 + \sqrt{3} \approx 2.7320$	$\beta^2 - 2\beta - 2$
54132	2.774622899	$\beta^4 - 2\beta^3 - 3\beta^2 + 2\beta + 1$
23154, 24315, 35421	2.831177207	$\beta^3 - 2\beta^2 - 2\beta - 1$
15423	2.879385242	$\beta^3 - 3\beta^2 + 1$
15324	2.912229178	$\beta^3 - 2\beta^2 - 3\beta + 1$
23145, 34251	3	$\beta - 3$
51423	3.234022893	$\beta^4 - 4\beta^3 + 3\beta^2 - 2\beta + 1$
32415, 43521	$\frac{3+\sqrt{13}}{2} \approx 3.3028$	$\beta^2 - 3\beta - 1$
15243	3.490863615	$\beta^3 - 3\beta^2 - 2\beta + 1$

The highest shift-complexity

For each n, the permutation

$$\rho = 1\,n\,2\,(n-1)\,3\,(n-2)\,\ldots$$

has the highest shift-complexity in $\mathcal{S}_n$.

The highest shift-complexity

For each n, the permutation

$$\rho = 1\,n\,2\,(n{-}1)\,3\,(n{-}2)\,\ldots$$

has the highest shift-complexity in $\mathcal{S}_n$.

$B(\rho)$ is the solution with $\beta > 1$ of

$$\beta = n - 2 + \frac{1}{\beta} + \frac{1}{\beta+1} - \frac{1}{\beta^{n-2-\delta}(\beta+1)}. \qquad (\delta{=}n \mod 2)$$

As n grows,

$$B(\rho) = n - 2 + \frac{2}{n} + O(\frac{1}{n^2}).$$

Thank you