Consecutive patterns in permutations and inversion sequences

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Consecutive patterns

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Example: 25134 avoids 132.

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- Permutations avoiding 123 and 321 are called alternating permutations, studied by André in the 19th century:

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Disregarding these disguised appearances, the systematic study of consecutive patterns in permutations started about 20 years ago.

Generating functions

For a fixed pattern σ, define the generating function

$$P_\sigma(u,z) = \sum_{n\geq 0} \sum_{\pi \in \mathcal{S}_n} u^{\#\{\text{occurrences of } \sigma \text{ in } \pi\}} \frac{z^n}{n!}.$$

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Example:

$$P_{21}(0,z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots = e^z$$

$$P_{21}(u,z) = 1 + z + (1+u)\frac{z^2}{2} + (u^2+4u+1)\frac{z^3}{6} + \dots$$

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- Classification of consecutive patterns into equivalence classes.

  (weak) Wilf-equivalence:

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- Asymptotic behavior and comparison of $|\mathcal{S}_n(\sigma)|$ for different patterns.

  Example: $|\mathcal{S}_n(132)| < |\mathcal{S}_n(123)|$ for $n \geq 4$.

Patterns of small length

Length 3: two strong Wilf-equivalence classes

$123 \overset{s}{\sim} 321$

$132 \overset{s}{\sim} 231 \overset{s}{\sim} 312 \overset{s}{\sim} 213$

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Length 4: seven strong Wilf-equivalence classes

$1234 \overset{s}{\sim} 4321$

$2413 \overset{s}{\sim} 3142$

$2143 \overset{s}{\sim} 3412$

$1324 \overset{s}{\sim} 4231$

$1423 \overset{s}{\sim} 3241 \overset{s}{\sim} 4132 \overset{s}{\sim} 2314$

$1342 \overset{s}{\sim} 2431 \overset{s}{\sim} 4213 \overset{s}{\sim} 3124 \overset{s}{\sim} 1432 \overset{s}{\sim} 2341 \overset{s}{\sim} 4123 \overset{s}{\sim} 3214$

$1243 \overset{s}{\sim} 3421 \overset{s}{\sim} 4312 \overset{s}{\sim} 2134$

All $\overset{s}{\sim}$ follow from reversal and complementation except for $\overset{s}{\sim}$.

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Length 4: seven strong Wilf-equivalence classes

$1234 \overset{s}{\sim} 4321$ — enumeration solved

$2413 \overset{s}{\sim} 3142$ — enumeration unsolved

$2143 \overset{s}{\sim} 3412$

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Conjecture (Nakamura '11)

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There are analogues of this conjecture in other settings, such as containment of words under the generalized factor order or patterns in inversion sequences.

Finding formulas for $P_\sigma(u, z)$

One method that we use to compute $P_\sigma(u, z)$ is an adaptation of the cluster method of Goulden and Jackson, based on inclusion-exclusion.

A k-cluster with respect to $\sigma \in \mathcal{S}_m$ is a permutation filled with k marked occurrences of σ that overlap with each other.

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Define the cluster generating function

$$C_\sigma(u,z) = \sum_{n,k} \#\{k\text{-clusters of length } n \text{ w.r.t. } \sigma\}\, u^k \frac{z^n}{n!}.$$

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$$P_\sigma(u,z) = \frac{1}{1 - z - C_\sigma(u-1,z)} \stackrel{\text{def}}{=} \frac{1}{\omega_\sigma(u,z)}.$$

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$\underline{\pi_1\pi_2\overline{\pi_3\pi_4\pi_5}}\overline{\pi_6\underline{\pi_7}}\underline{\pi_8\pi_9\pi_{10}\pi_{11}}$ is a cluster w.r.t. $\sigma = 14253$

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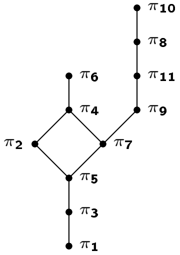

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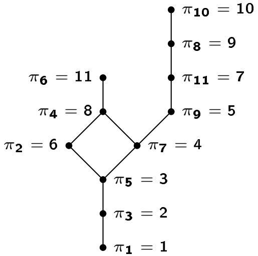

Ex: $\underline{1\ 6\ \overline{2\ 8\ 3}}\ \overline{11}\ \underline{\overline{4}\ 9\ 5\ 10\ 7}$

Monotone and related patterns

Theorem (E.–Noy '01)

For $\sigma = 12\dots m$, $\omega_\sigma(u,z)$ is the solution of

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When $u = 0$, we have

$$\omega_{12\ldots m}(0,z) = \sum_{j\geq 0} \left(\frac{z^{jm}}{(jm)!} - \frac{z^{jm+1}}{(jm+1)!} \right).$$

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More generally, we get similar differential equations for any σ for which all its cluster posets are chains, such as

$$\sigma = 12\dots(s-1)(s+1)s(s+2)(s+3)\dots m.$$

Non-overlapping patterns

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Similar arguments give differential equations for $\sigma = 12534$ and $\sigma = 13254$, which aren't non-overlapping.

The pattern 1324

Theorem (E.–Noy, Liese–Remmel, Dotsenko–Khoroshkin '11)

For $\sigma = 1324$, $\omega_\sigma(u,z)$ *is the solution of*

$$z\omega^{(5)} - ((u-1)z-3)\omega^{(4)} - 3(u-1)(2z+1)\omega^{(3)} + (u-1)((4u-5)z-6)\omega'' + (u-1)(8(u-1)z-3)\omega' + 4(u-1)^2 z\omega = 0$$

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The construction generalizes to patterns of the form

$$\sigma = 134\dots(s+1)2(s+2)(s+3)\dots m.$$

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Theorem (Beaton–Conway–Guttmann '18, conjectured by E.–Noy '11)

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Other patterns of length 4

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There is an analogous question in the case of "classical" (i.e. non-consecutive) patterns.
Garrabrant–Pak '15 show that some generating functions for permutations avoiding sets of classical patterns are not D-finite.

Asymptotic behavior

Theorem (E. '05)

For every σ, the limit

$$\rho_\sigma := \lim_{n\to\infty} \left(\frac{|\mathcal{S}_n(\sigma)|}{n!} \right)^{1/n} \qquad \textit{exists.}$$

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Theorem (Ehrenborg–Kitaev–Perry '11)

For every σ,

$$\frac{|\mathcal{S}_n(\sigma)|}{n!} = \gamma_\sigma \rho_\sigma^n + O(\delta^n),$$

for some constants γ_σ and $\delta < \rho_\sigma$.

The proof uses methods from spectral theory.

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For what pattern $\sigma \in \mathcal{S}_m$ is $|\mathcal{S}_n(\sigma)|$ largest?

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Interestingly, the analogous result for classical (i.e. non-consecutive) patterns is false; it is not known what the most avoided pattern is.

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Again, there is no analogous known result for classical (i.e. non-consecutive) patterns.

Consecutive patterns in inversion sequences

(joint with Juan Auli)

Inversion sequences

An inversion sequence of length n is an integer sequence $e = e_1 e_2 \cdots e_n$ such that $0 \leq e_i < i$.

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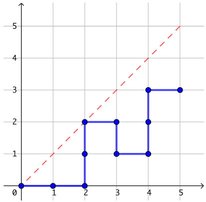
Example. $e = 00213 \in \mathbf{I}_5$.



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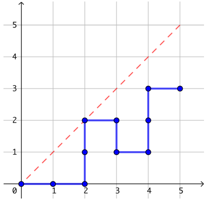

Permutations can be encoded as inversion sequences via the bijection $\Theta : \mathcal{S}_n \to \mathbf{I}_n$, defined by $\Theta(\pi) = e_1 e_2 \cdots e_n$ where

$$e_j = |\{i : i < j \text{ and } \pi_i > \pi_j\}|.$$

For instance, $\Theta(35142) = 00213$.

Consecutive patterns in inversion sequences

An occurrence of the (consecutive) pattern $p = p_1 p_2 \cdots p_l$ in an inversion sequence $e \in \mathbf{I}_n$ is a subsequence of adjacent entries $e_i e_{i+1} \cdots e_{i+l-1}$ in the same relative order as p.

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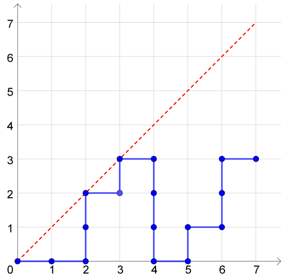

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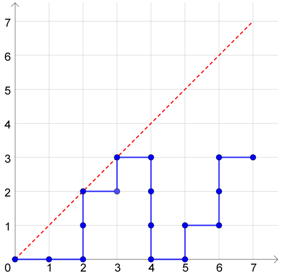

Let $\mathbf{I}_n(p) = \{e \in \mathbf{I}_n : e \text{ avoids } p\}$.

Avoiding consecutive patterns of length 3

We have formulas or recurrences for the numbers $|\mathbf{I}_n(p)|$ for all 13 patterns p of length 3.

Proposition (Auli–E. '19)

$$|\mathbf{I}_n(000)| = \frac{(n+1)! - d_{n+1}}{n},$$

where d_n is the number of derangements in $\mathcal{S}_n$.

Equivalences between patterns

For $e \in \mathbf{I}_n$ and a consecutive pattern p, let

$$\mathrm{Oc}(p, e) = \{i : e_i e_{i+1} e_{i+2} \text{ is an occurence of } p\}.$$

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Note that $p \overset{ss}{\sim} p' \Rightarrow p \overset{s}{\sim} p' \Rightarrow p \overset{w}{\sim} p'$.

Equivalences between patterns of length 3

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The only equivalence for patterns of length 3 is

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Corollary (conjectured by Baxter–Pudwell '12, proved non-bijectively by Baxter–Shattuck and Kasraoui)

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We can prove this with a sequence of bijections:
$$\mathcal{S}_n(124\text{—}3) \leftrightarrow \mathbf{I}_n(100) \cap \mathbf{I}_n(210) \leftrightarrow \mathbf{I}_n(110) \cap \mathbf{I}_n(210) \leftrightarrow \mathcal{S}_n(421\text{—}3).$$

Patterns of length 4

Theorem (Auli–E.)

Here are all equivalences between consecutive patterns of length 4:

- $0102 \overset{ss}{\sim} 0112$
- $0021 \overset{ss}{\sim} 0121$
- $1002 \overset{ss}{\sim} 1012 \overset{ss}{\sim} 1102$
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Consecutive patterns in dynamical systems

Deterministic or random?

Two sequences of numbers in $[0,1]$:

.6416, .9198, .2951, .8320, .5590, .9861, .0550, .2078, .6584, .8996, .3612, .9230, .2844, .8141, .6054, .9556, .1687, .5637, . . .

.9129, .5257, .4475, .9815, .4134, .9930, .1576, .8825, .3391, .0659, .1195, .5742, .1507, .5534, .0828, .3957, .1886, .0534, . . .

Which one is random? Which one is deterministic?

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The first one is deterministic: taking $f(x) = 4x(1-x)$, we have

$f(.6146) = .9198,$
$f(.9198) = .2951,$
$f(.2951) = .8320,$
$\dots$

Allowed patterns of a map

Let X be a linearly ordered set, $f : X \to X$. For each $x \in X$ and $n \geq 1$, consider the sequence

$$x, f(x), f^2(x), \ldots, f^{n-1}(x).$$

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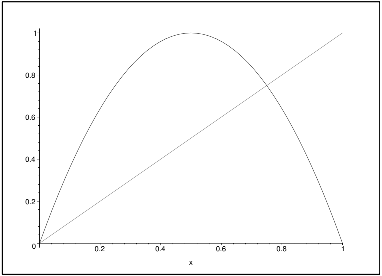
If there are no repetitions, the relative order of the entries determines a permutation, called an allowed pattern of f.

Example

$$\begin{array}{rccl} f: & [0,1] & \to & [0,1] \\ & x & \mapsto & 4x(1-x). \end{array}$$



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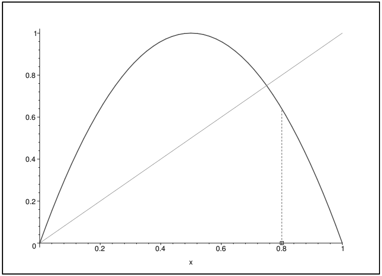

For $x = 0.8$ and $n = 4$, the sequence

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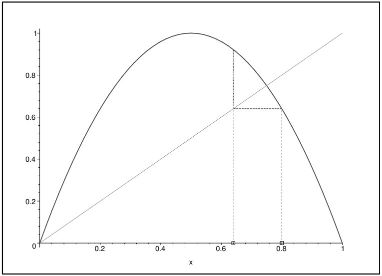

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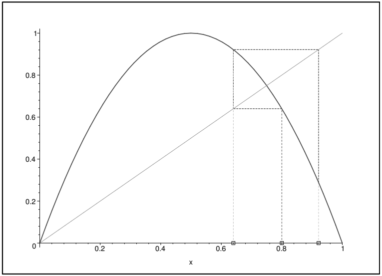

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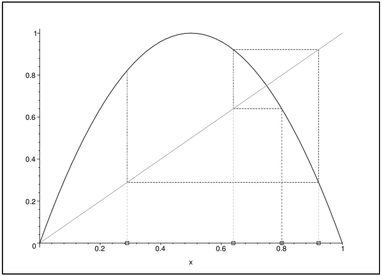

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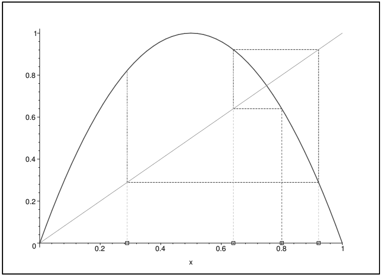

For $x = 0.8$ and $n = 4$, the sequence

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determines the permutation 3241, so it is an allowed pattern.

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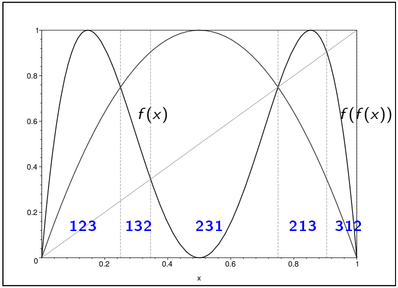
Thus, Allow(f) can be characterized by avoidance of a (possibly infinite) set of consecutive patterns.

The permutations not in Allow(f) are called forbidden patterns of f.

Example: $f(x) = 4x(1-x)$

Taking different $x \in [0,1]$, the patterns 123, 132, 231, 213, 312 are realized. However, 321 is a forbidden pattern of f.



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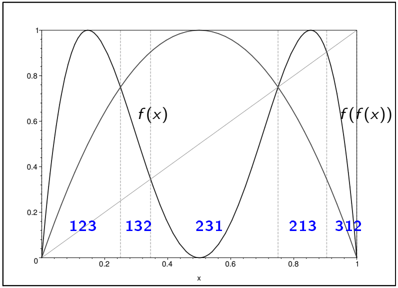

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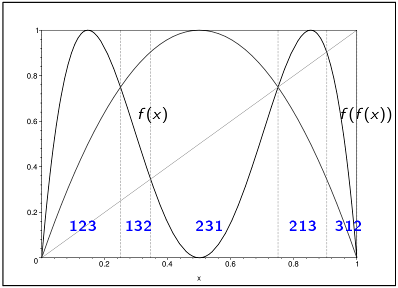

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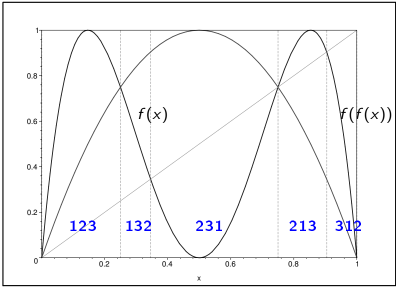

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Theorem (E.–Liu '11): $f(x) = 4x(1-x)$ on the unit interval has infinitely many basic forbidden patterns.

Forbidden patterns

Let $I \subset \mathbb{R}$ be a closed interval.

Theorem (Bandt–Keller–Pompe '02)

Let $f : I \to I$ be a piecewise monotone map. Then

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Provides a combinatorial way to compute the topological entropy, which is a measure of the complexity of the dynamical system.

Deterministic vs. random sequences

Back to the original sequence:

$$.6416,\ .9198,\ .2951,\ .8320,\ .5590,\ .9861,\ .0550,\ .2078,\ .6584,\ .8996,\ .3612,\ .9230,\ .2844,\ .8141,\ .6054,\ .9556,\ .1687,\ .5637,\dots$$

We see that the pattern 321 is missing from it. This is because $x_{i+1} = f(x_i)$ with $f(x) = 4x(1-x)$.

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If it was a random sequence, any pattern would eventually appear.

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- What sets of permutations are of the form Allow(f) for some f?
- Design pattern-based tests to distinguish random sequences from deterministic ones.

Thank you