Sorting by placement and shift

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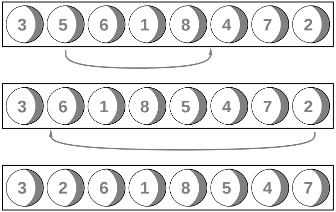
The *homing* algorithm

Given a permutation $\pi \in S_n$, repeat the following *placement* step:

- Choose an entry $\pi(i)$ such that $\pi(i) \neq i$.
- Place $\pi(i)$ in the correct position.
- Shift the other entries as necessary.



Main questions

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- How many steps does it take in the worst case...
  - with a good choice of placements?
  - with a random choice of placements?
  - with a bad choice of placements?

Fast Homing: Well-chosen placements

Theorem

An algorithm that always places the smallest or largest available number will terminate in at most $n-1$ steps.

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Let k be the length of the longest increasing subsequence in π. Then no sequence of fewer than $n-k$ placements can sort π.

Random placements

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The expected number of steps required by random homing from $\pi \in S_n$ *is at most* $\frac{n^2+n-2}{4}$.

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- Suppose that we have a permutation where k of the extremal numbers are home:

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- With probability $\leq \frac{2}{n-k}$, the next step will place an additional extremal number.

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- Suppose that we have a permutation where k of the extremal numbers are home:

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- With probability $\leq \frac{2}{n-k}$, the next step will place an additional extremal number.
- Total expected number of steps is $\leq \sum_{k=0}^{n-2} \frac{n-k}{2}$.

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Slow Homing: Example

Starting from

$$2\,3\,4\,5\,6\,7\dots n\,1$$

place always the leftmost possible entry:

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It takes $2^{n-1}-1$ steps to sort this permutation.

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To prove this, consider the reverse algorithm.
We will show that, starting from the identity permutation, one can perform at most $2^{n-1}-1$ *displacements*.

$$2^{n-1}-1 = \underbrace{2^{n-2}}_{\text{until 1 and } n \text{ are displaced}} + \underbrace{2^{n-2}-1}_{\text{after displacing 1 and } n}$$

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After 2^{n-2} *displacements, both* 1 *and* n *have been displaced and will never be displaced again.*

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Proof.

- Note that 1 and n can each be displaced only once.
- If after 2^{n-2} displacements one of these values hasn't been displaced, then it played no role in the process.
- Hence the remaining $n-1$ numbers allowed more than $2^{n-2}-1$ steps, contradicting the induction hypothesis.

□

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Assign to each permutation π a code $\alpha(\pi) = \alpha_2\alpha_3\ldots\alpha_{n-1}$, where

$$\alpha_i = \left\{ \begin{array}{c} 0 \\ + \\ - \end{array} \right\} \text{ if entry } i \text{ is } \left\{ \begin{array}{c} \text{exactly} \\ \text{to the right of} \\ \text{to the left of} \end{array} \right\} \text{ home.}$$

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$$\pi = 3\,5\,6\,1\,8\,4\,7\,2 \quad \longrightarrow \quad \alpha(\pi) = + - + - - 0$$

The weight of a code

$$\alpha = \quad + \quad - \quad + \quad - \quad - \quad 0$$

Define the weight of a code α recursively:

The weight of a code

$$\alpha = \begin{matrix} + & - & + & - & - & 0 \\ 5 & 1 & 3 & 3 & 4 & \end{matrix}$$

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- For each $-$, count the number of symbols to its left, and for each $+$, count the number of symbols to its right.

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- Let *d* be the largest of these numbers, and let $\hat{\alpha}$ be the code obtained by deleting the corresponding symbol.

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- Define
$$w(\alpha) = 2^d + w(\hat{\alpha}).$$

The weight of a code: example

$$w(\ + \ - \ + \ - \ - \ 0\ )$$

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Bound on the weight

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Proof.

In the recursion,

$$w(\alpha) \leq 2^{k-1} + w(\hat{\alpha}),$$

with equality when a $-$ is deleted from the right or a $+$ from the left. □

The weight increases at each displacement

Lemma

Let $\pi \in S_n$ *with* $\pi(1) \neq 1$ *and* $\pi(n) \neq n$*, and let* π' *be the result of applying some displacement to* π*. Let* $\alpha = \alpha(\pi)$ *and* $\alpha' = \alpha(\pi')$*. Then*

$$w(\alpha') > w(\alpha).$$

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- Similarly if i is displaced to the right.

Finishing the proof

Combining these lemmas, the maximum number of displacements is

- at most 2^{n-2} until 1 and n are displaced, plus
- at most $2^{n-2}-1$ after 1 and n have been displaced.

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- at most $2^{n-2}-1$ after 1 and n have been displaced.

So at most $2^{n-1}-1$ in total.

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where B_n = *n-th Bell number* = # *partitions of* $\{1, 2, \ldots, n\}$.

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B_n grows super-exponentially.

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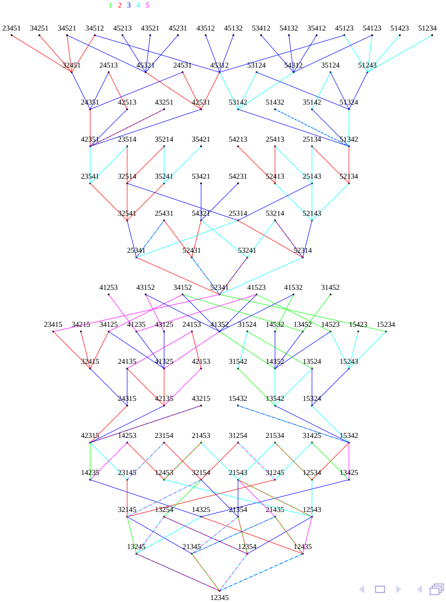
Theorem

$$F(u,v) = uv + uv\frac{\partial}{\partial u}F(u,v) + uv\frac{\partial}{\partial v}F(u,v) - u^2v^2\frac{\partial^2}{\partial u \partial v}F(u,v)$$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THANK YOU