An involution on lattice paths between two boundaries

Sergi Elizalde*

Abstract

We give an involution on the set of lattice paths from $(0,0)$ to (a,b) with steps $N=(0,1)$ and $E=(1,0)$ that lie between two boundaries T and B, which proves that the statistics 'number of E steps shared with T' and 'number of E steps shared with B' have a symmetric joint distribution on this set. This generalizes a result of Deutsch for the case of Dyck paths.

1 Introduction

All the paths in this note are lattice paths in $\mathbb{Z}^2$ with steps $N=(0,1)$ and $E=(1,0)$, unless otherwise stated. Fix positive integers a,b, and let T and B be two paths from $(0,0)$ to (a,b). We assume that T lies weakly above B (equivalently, B lies weakly below T), by which we mean that no point of T is strictly below and to the right of any point of B. Let $\mathcal{P}(T,B)$ be the set of paths from $(0,0)$ to (a,b) that lie between T and B (that is, weakly below T and weakly above B). For each path $P\in\mathcal{P}(T,B)$, define the statistics

$$\begin{aligned} t(P) &= \text{number of } E \text{ steps where } P \text{ and } T \text{ coincide},\\ b(P) &= \text{number of } E \text{ steps where } P \text{ and } B \text{ coincide}. \end{aligned}$$

The purpose of this note is to prove that the joint distribution of the statistics (t,b) is symmetric over the set $\mathcal{P}(T,B)$, which can be stated as follows.

Theorem 1. *For any given paths T and B as above,*

$$\sum_{P\in\mathcal{P}(T,B)} u^{t(P)}v^{b(P)} = \sum_{P\in\mathcal{P}(T,B)} u^{b(P)}v^{t(P)}.$$

For example, if $T=NNENEE$ and $B=ENEENN$, then the above polynomial is $u^3+u^2v+uv^2+v^3+2u^2+2uv+2v^2+2u+2v+1$. To prove Theorem 1, we give an involution $\psi:\mathcal{P}(T,B)\to\mathcal{P}(T,B)$ with the property that for all $P\in\mathcal{P}(T,B)$, $t(\psi(P))=b(P)$ and $b(\psi(P))=t(P)$.

2 The involution

The involution ψ is defined by first splitting the path P into pieces, on which one applies a transformation φ, which is in turn described in terms of a simple map f.

*Department of Mathematics, Dartmouth College, Hanover, NH 03755. E-mail: `sergi.elizade@dartmouth.edu`

2.1 Splitting the path

First of all, note that if T and B share vertices (lattice points) other than $(0,0)$ and (a,b), we can split T and B into blocks, so that each block from T and the corresponding block from B agree either everywhere or only on their first and last vertex. Since P passes though all the vertices that T and B have in common, those vertices of P are unchanged by ψ. It is enough then to define ψ on blocks of T and B that only agree on their first and last vertex. Thus, we will assume from now on that T and B only intersect at $(0,0)$ and (a,b), and that $n = a + b \geq 2$.

Denote by $\alpha_0, \alpha_1, \ldots, \alpha_n$ the vertices of P in order from $\alpha_0 = (0,0)$ to $\alpha_n = (a,b)$. For $i < j$, denote by $P[i,j]$ the fragment of P between α_i and α_j. For $0 < i < n$, each vertex α_i can be on T or on B, but not on both. Let Z be set of pairs (i,j), with $0 < i < j < n$, such that α_i is on T, α_j is on B, and for $i < \ell < j$, α_ℓ is on neither T nor B. If Z is not empty, suppose that $Z = \{(i_1, j_1), \ldots, (i_{s-1}, j_{s-1})\}$, where $i_1 < \cdots < i_{s-1}$. Note that the intervals $[i_k, j_k]$ are disjoint for all k, and that the fragments $R_k = P[i_k, j_k]$ have no E steps in common with neither T nor B. With the convention $j_0 = 0$ and $i_s = n$, let $Q_k = P[j_{k-1}, i_k]$ for $1 \leq k \leq s$, so P can be broken down as

$$P \;=\; Q_1\, R_1\, Q_2\, R_2\, Q_3\, \ldots\, R_{s-1}\, Q_s.$$

Defining $t(Q_k)$ (resp. $b(Q_k)$) as the number of E steps where Q_k and T (resp. Q_k and B) coincide, it is clear that $t(P) = t(Q_1) + \cdots + t(Q_s)$ and $b(P) = b(Q_1) + \cdots + b(Q_s)$. Figure 1 shows an example of this decomposition.

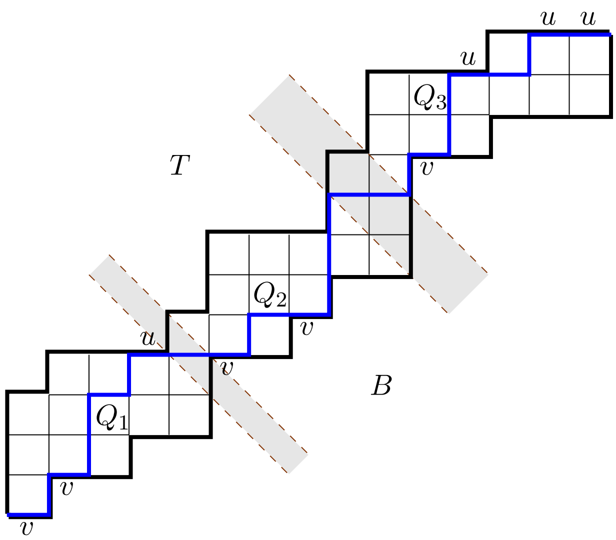

Figure 1: A path P lying between T and B. The dotted lines indicate the cuts at i_1, j_1, i_2, j_2. The E steps in common with T and B are marked with u and v, respectively.

We construct $\psi(P)$ by leaving R_k unchanged for all k, and by applying a certain transformation φ to each of the Q_k, that is,

$$\psi(P) \;=\; \varphi(Q_1)\, R_1\, \varphi(Q_2)\, R_2\, \varphi(Q_3)\, \ldots\, R_{s-1}\, \varphi(Q_s). \tag{1}$$

2.2 Definition of φ

Let $\mathcal{Q}(T,B)$ be the set of paths that start at a point on B (possibly $(0,0)$), end at a point on T (possibly (a,b)), lie between T and B, and, excluding $(0,0)$ and (a,b), have no intersection with T preceding an intersection with B. We will define an involution φ on $\mathcal{Q}(T,B)$ that preserves the first and last vertex, and satisfies $t(\varphi(Q)) = b(Q)$ and $b(\varphi(Q)) = t(Q)$. Note that, by construction, $Q_k \in \mathcal{Q}(T,B)$ for all k.

For $Q \in \mathcal{Q}(T,B)$, let $t = t(Q)$ and $b = b(Q)$ be the number of E steps that Q has in common with T and with B, respectively. Note that the E steps shared with B are to the left of those shared with T. We define

$$\varphi(Q) = f^{b-t}(Q), \tag{2}$$

where f and f^{-1} are defined in the next subsection.

2.3 Definition of f and f^{-1}

Let $Q \in \mathcal{Q}(T,B)$ be such that $b(Q) > 0$. Then Q can be uniquely broken down as $Q = X\bar{E}YZ$, where $\bar{E}$ is the rightmost E step where Q and B coincide, and the point between Y and Z is the first vertex $\alpha \neq (0,0)$ where Q and T intersect. Note that α exists because Q ends at a point on T. Note also that Y begins and ends with a N step, whereas the step of T ending at α is an E step. To construct $f(Q)$, move $\bar{E}$ and put it right after Y, that is,

$$f(Q) = XY\bar{E}Z.$$

Clearly, $f(Q)$ lies weakly above B and, by the choice of α, $f(Q)$ also lies weakly below T. The fragment Y in Q intersects T only at the last point, and the fragment Y in $f(Q)$ intersects B only at the first point. In both Q and $f(Q)$, Y has no E steps in common with either B or T. Since $\bar{E}$ was a common step of Q and B that became a common step of $f(Q)$ and T, we see that $t(f(Q)) = t(Q) + 1$ and $b(f(Q)) = b(Q) - 1$. We remark that the last intersection of $f(Q)$ with B (not counting (a,b)) is the vertex between X and Y. Since X has no intersections with T (not counting $(0,0)$), we have that $f(Q) \in \mathcal{Q}(T,B)$. Examples of the map f are given in Figure 2.

The definition of f^{-1} is analogous to that of f if we rotate the paths T, B and Q by 180 degrees. We describe it here for convenience, and we argue that it is indeed the inverse of f. Let now $Q \in \mathcal{Q}(T,B)$ be such that $t(Q) > 0$. Then Q can be uniquely broken down as $Q = XY\bar{E}Z$, where $\bar{E}$ is the leftmost E step where Q and T coincide, and the point between X and Y is the last vertex $\beta \neq (a,b)$ where Q and B intersect. To construct $f^{-1}(Q)$, move $\bar{E}$ and put it right before Y, that is,

$$f^{-1}(Q) = X\bar{E}YZ.$$

As before, we have that $f^{-1}(Q) \in \mathcal{Q}(T,B)$.

To see that f^{-1} is indeed the inverse of f, note that when f is applied to $Q = X\bar{E}YZ$ to produce $f(Q) = XY\bar{E}Z$, the moved step $\bar{E}$ becomes the leftmost E step where $f(Q)$ and T coincide. Since the last intersection of $f(Q)$ with B (not counting (a,b)) is the vertex between X and Y, f^{-1} moves $\bar{E}$ back to its original position when applied to $f(Q)$. So, if $b(Q) > 0$, then $f^{-1}(f(Q)) = Q$. Similarly, the step of Q moved when applying f^{-1} becomes the rightmost E step where $f^{-1}(Q)$ and B coincide, which is moved back to its original position by f. So, if $t(Q) > 0$, then $f(f^{-1}(Q)) = Q$. It also follows that $t(f^{-1}(Q)) = t(Q) - 1$ and $b(f^{-1}(Q)) = b(Q) + 1$.

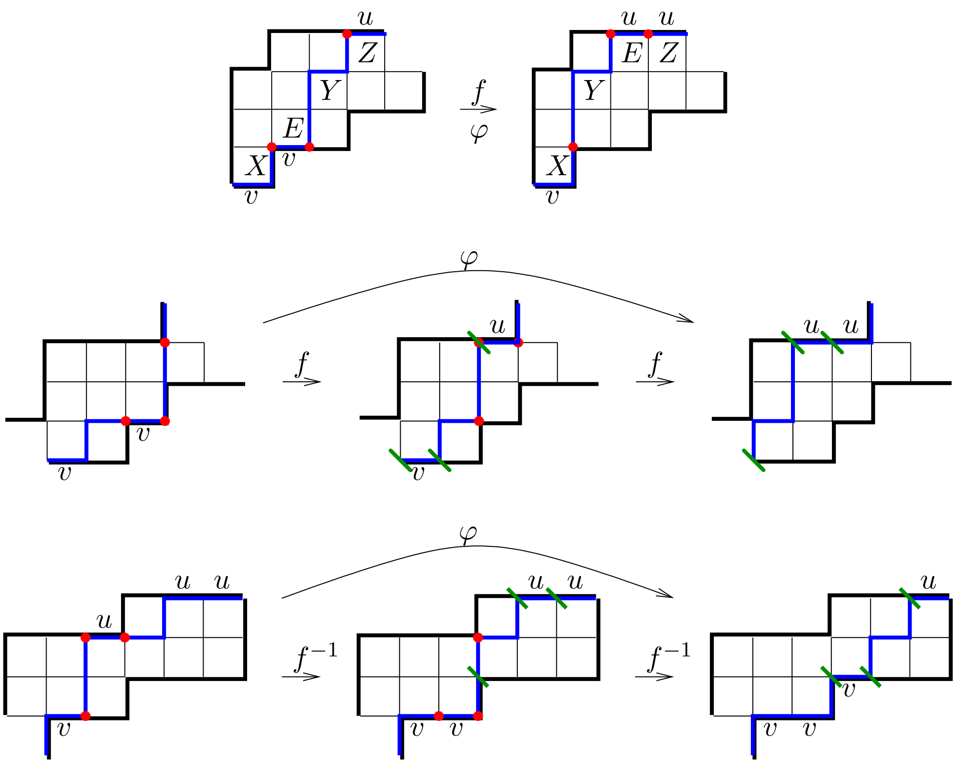

Figure 2: The map φ applied to the pieces of the path in Figure 1.

From definition (2) and the properties of f and f^{-1}, it is clear that $\varphi(Q) \in \mathcal{Q}(T, B)$ and $t(\varphi(Q)) = t + (b - t) = b$ and $b(\varphi(Q)) = b - (b - t) = t$. Thus, we see from definition (1) that $\psi(P)$ lies between T and B and satisfies $t(\psi(P)) = b(P)$ and $b(\psi(P)) = t(P)$. Finally, we have that $\varphi(\varphi(Q)) = f^{t-b}(f^{b-t}(Q)) = Q$, so φ is an involution. Since the spitting of $\psi(P)$ in order to apply ψ occurs at the same points where the splitting of P took place, ψ is an involution as well. The image under ψ of the path in Figure 1 is shown in Figure 3.

3 Connections to work in the literature

3.1 Dyck paths and generalizations

In the particular case that $T = N^m E^m$ and $B = (NE)^m$, $\mathcal{P}(T, B)$ is the set of Dyck paths of semilength m. For $P \in \mathcal{P}(T, B)$, $t(P)$ is the number of E steps at the end of P, and $b(P)$ is the number of returns to the diagonal $y = x$. When drawing Dyck paths as paths with steps $U = (1, 1)$ and $D = (1, -1)$ starting at the origin and not going below the x-axis, these statistics correspond to the height of the last peak and the number of returns to the x-axis, respectively. A bijective proof of the fact that these statistics are equidistributed on Dyck paths was given by Deutsch [3], who later also gave an involution [4] proving the symmetry of their joint distribution. Our involution ψ, when restricted to the case of Dyck paths, is quite different from Deutsch's involution, which is defined recursively and does not seem to extend to the general case. Additionally, our map f can be used to prove the following statement.

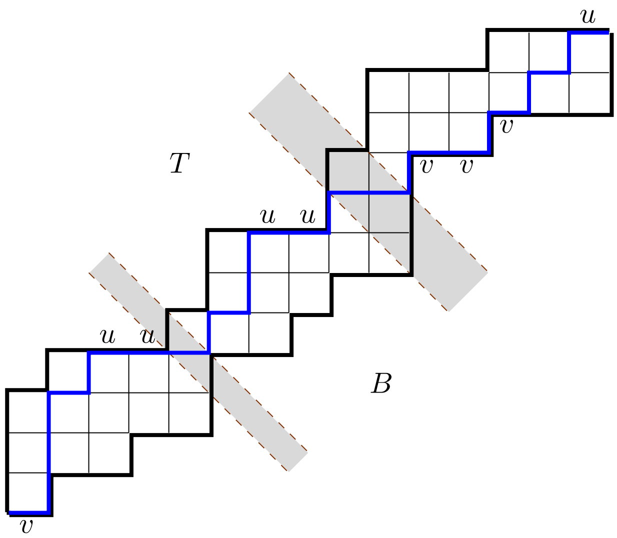

Figure 3: The path $\psi(P)$, where P is the path in Figure 1.

Proposition 2. *Fix positive integers a, b. Let $T = N^b E^a$, and let B be a path from $(0,0)$ to (a,b) that intersects T only at these two points. Then, for all $i, j \geq 0$, the coefficient of $u^i v^j$ in the polynomial*

$$\sum_{P \in \mathcal{P}(T,B)} u^{t(P)} v^{b(P)}$$

depends only on $i + j$.

Proof. When applying ψ to a path $P \in \mathcal{P}(T,B)$, P gets decomposed either as $P = Q_1$ or as $P = Q_1 R_1 Q_2$. In the second case, Q_1 consists only of N steps, so $Q_1 R_1$ has no E steps in common with either T or B. Let us define $f(P) = Q_1 R_1 f(Q_2)$ in this case, which is also equivalent to applying the definition of f from Section 2.3 directly to P (the definition works even if $P \notin \mathcal{Q}(T,B)$). Similarly, let $f^{-1}(P) = Q_1 R_1 f^{-1}(Q_2)$. We now have a sequence of bijections

$$\{P \in \mathcal{P}(T,B) : t(P) = 0, b(P) = i+j\} \xrightarrow{f} \{P \in \mathcal{P}(T,B) : t(P) = 1, b(P) = i+j-1\} \xrightarrow{f} \cdots$$
$$\cdots \xrightarrow{f} \{P \in \mathcal{P}(T,B) : t(P) = i+j, b(P) = 0\}$$

(see Figure 4), from where the statement follows. □

When $T = N^m E^m$ and $B = (EN)^m$, the above proposition translates into the fact that the number of Dyck paths of fixed semilength $m+1$ with $j+1$ returns whose last peak has height $i+1$ depends only on the sum $i+j$. In particular, it equals the number of paths $P \in \mathcal{P}(T,B)$ with $t(P) = i+j$ and $b(P) = 0$. By deleting the NE^{i+j} at the end, such paths are in bijection with paths from $(0,0)$ to $(m-i-j, m-1)$ not going under $y = x$, which are counted by the ballot number

$$\frac{i+j}{2m-i-j}\binom{2m-i-j}{m}.$$

Let us mention that for the case of Dyck paths, the symmetry of the joint distribution of the pair (t, b) can also be proved using standard generating function techniques, based on the usual

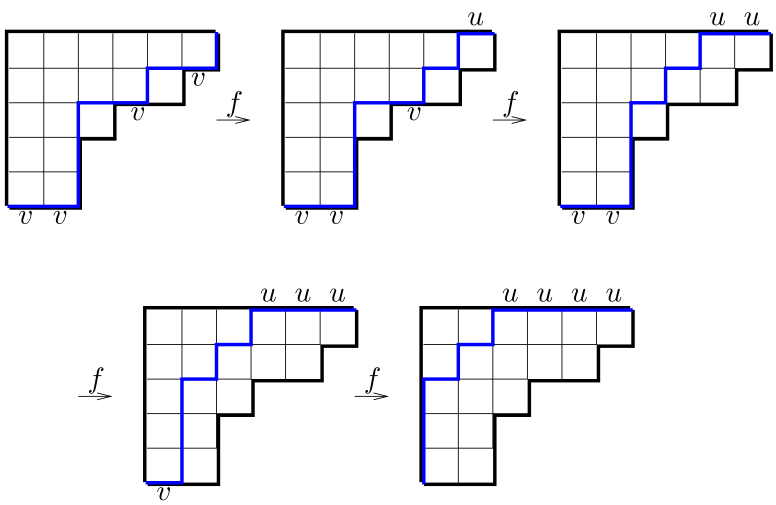

Figure 4: A sequence of bijections on paths with $t(P) + b(P) = 4$.

recursive decomposition of Dyck paths. However, these techniques fail in the general setting of Theorem 1, hence the need for our bijective proof.

3.2 Lattice path matroids

In [2], Bonin, de Mier and Noy defined the following type of transversal matroids. As before, fix two lattice paths T and B from $(0,0)$ to (a,b) with T weakly above B. For each $P \in \mathcal{P}(T,B)$, let $N_P \subseteq [a+b]$ be the set of positions of the N steps of P when viewed as a word in $\{N,E\}^{a+b}$. Then $\{N_P : P \in \mathcal{P}(T,B)\}$ is the set of bases of a matroid $M[T,B]$, called a *lattice path matroid.* In the particular case that $T = N^m E^m$ and $B = (EN)^m$, $M[T,B]$ was called a *Catalan matroid* by Ardila [1]. In [2, Theorem 5.4], it is shown that the external activity of the basis N_P with respect to the order $1 < 2 < \cdots < a+b$ on the ground set is precisely $b(P)$. Similarly, $t(B)$ is the external activity of N_P with respect to the order $1 > 2 > \cdots > a+b$. With this interpretation, the equidistribution of the statistics t and b over $\mathcal{P}(T,B)$ follows from the fact that the Tutte polynomial of $M[T,B]$, when expressed as a sum of bases weighted by their internal and external activities, is independent of the linear order on the ground set. On the other hand, the symmetry of the joint distribution (t,b) stated in Theorem 1 does not appear to be a consequence of any straightforward matroidal property.

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References

[1] F. Ardila, The Catalan matroid, *J. Combin. Theory Ser. A* 104 (2003), 49–62.

[2] J. Bonin, A. de Mier, M. Noy, Lattice path matroids: enumerative aspects and Tutte polynomials, *J. Combin. Theory Ser. A* 104 (2003), 63–94.

[3] E. Deutsch, A bijection on Dyck paths and its consequences, *Discrete Math.* 179 (1998), 253–256.

[4] E. Deutsch, An involution on Dyck paths and its consequences, *Discrete Math.* 204 (1999), 163–166.