Schur-positive grid classes and cyclic descents of SYT

Sergi Elizalde

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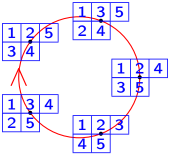
Joint work with Ron Adin and Yuval Roichman



Oberwolfach, May 2018

Permutations and quasisymmetric functions

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Example: For $\pi = 132$, $\mathrm{Des}(\pi) = \{2\}$ and

$$F_{132} = \sum_{i_1 \le i_2 < i_3} x_{i_1} x_{i_2} x_{i_3} = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + \cdots + x_1 x_2 x_3 + x_1 x_2 x_4 + \dots.$$

Quasisymmetric: coeff of $x_{i_1}^{\alpha_1} \dots x_{i_k}^{\alpha_k}$ is the same for any $i_1 < \cdots < i_k$.

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Define $\mathcal{Q}(A)$ similarly if A is a multiset.

Known Schur-positive sets

- [Gessel '84]: $\mathcal{S}_n$. $$\mathcal{Q}(\mathcal{S}_n) = \sum_{\lambda \vdash n} |\operatorname{SYT}(\lambda)|\, s_\lambda.$$

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- [Gessel–Reutenauer '93]: Subsets of $\mathcal{S}_n$ closed under conjugation. In particular,
  - involutions,
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- [Adin–Roichman '15]: Sets of the form $\{\pi \in \mathcal{S}_n : \mathrm{inv}(\pi) = k\}$.

A new Schur-positive set

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$\mathcal{A}_n$ is Schur-positive, and

$$\mathcal{Q}(\mathcal{A}_n) = s_n + s_{1^n} + \sum_{k=2}^{n-2} s_{n-k,2,1^{k-2}} + 2\sum_{k=1}^{n-2} s_{n-k,1^k}.$$

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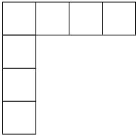
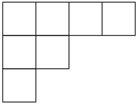
The proof constructs a Des-preserving bijection between $\mathcal{A}_n$ and SYT of certain shapes.





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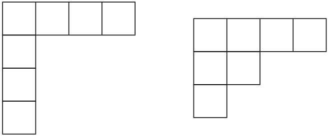

Incidentally,
$\mathcal{A}_n = \mathcal{S}_n(1324, 1342, 2413, 2431, 3124, 3142, 4213, 4231)$.

Geometric grid classes

Let M be a $\{0,1,-1\}$-matrix.

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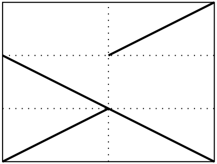
$\Gamma(M) =$ 

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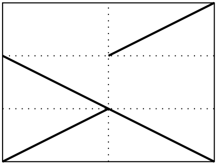
$\Gamma(M) =$ 

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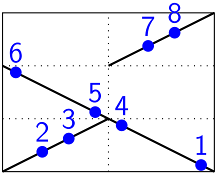

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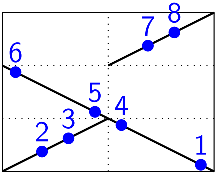

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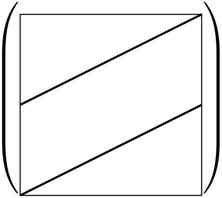
Theorem (Albert, Atkinson, Bouvel, Ruškuc, Vatter '13)

Every geometric grid class can be characterized by avoidance of a finite set of patterns.

Examples of geometric grid classes

$\mathcal{G}_n$  $= \mathcal{S}_n(321, 2143, 2413).$

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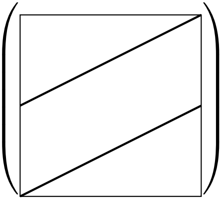
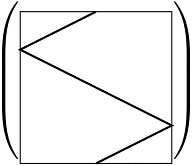
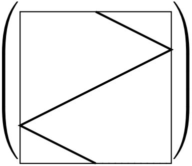
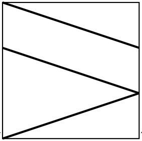
$\mathcal{G}_n$  $= \mathcal{S}_n(321, 2143, 2413)$.

Arc permutations can be expressed as a union of two geometric grid classes:

$\mathcal{A}_n = \mathcal{G}_n$  $\cup\ \mathcal{G}_n$  .

Schur-positive geometric grid classes

[E.-Roichman '15]: One-column grid classes are Schur-positive.

$\mathcal{Q}\Bigg(\mathcal{G}_5\Bigg($  $\Bigg)\Bigg) = s_5 + 2\, s_{4,1} + 2\, s_{3,2} + 3\, s_{3,1^2} + 4\, s_{2^2,1} + 4\, s_{2,1^3} + s_{1^5}\,.$

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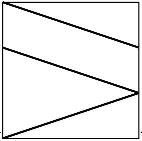
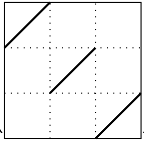
$\mathcal{Q}\Bigg(\mathcal{G}_5\Bigg($  $\Bigg)\Bigg) = s_5+2\,s_{4,1}+2\,s_{3,2}+3\,s_{3,1^2}+4\,s_{2^2,1}+4\,s_{2,1^3}+s_{1^5}\,.$

[E.-Roichman '15]: Layered permutations are Schur-positive.

$\mathcal{Q}\Bigg(\mathcal{G}_n\Bigg($  $\Bigg)\Bigg) = s_n + s_{n-1,1} + s_{n-2,1^2}\,.$

Vertical rotations

Let $c \in \mathcal{S}_n$ be the n-cycle $c = (1, 2, \ldots, n)$, and let $C_n = \langle c \rangle = \{c^k : \ 0 \leq k < n\}$ be the subgroup it generates.

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For $A \subseteq \mathcal{S}_n$, $C_n A$ is the multiset of vertical rotations of elements in A.

Theorem (E.-Roichman '15)

For a one-column grid class $\mathcal{H}_n$, the multiset $C_n \mathcal{H}_n$ is Schur-positive.

Arc permutations revisited

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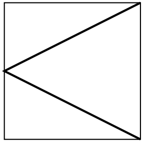
$C_n \times$ 

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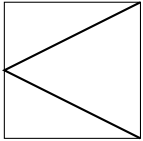
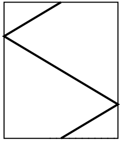
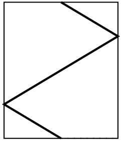
$C_n \times$  $= 2$  $+ 2$ 

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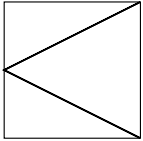
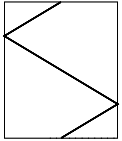
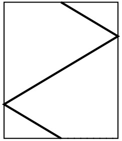
$C_n \times$  $= 2$  $+ 2$  $= 2\mathcal{A}_n.$

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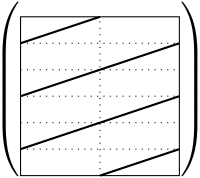
For example, $\mathcal{G}_n$  is Schur-positive.

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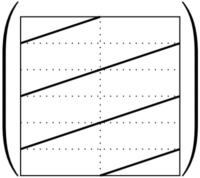
For example, $\mathcal{G}_n$  is Schur-positive.

As a byproduct of the proof, we get a notion of cyclic descents on SYT of certain shapes.

Cyclic descents of permutations

The cyclic descent set of $\pi \in \mathcal{S}_n$ is

$$\mathrm{cDes}(\pi) = \begin{cases} \mathrm{Des}(\pi) \cup \{n\} & \text{if } \pi_n > \pi_1, \\ \mathrm{Des}(\pi) & \text{otherwise.} \end{cases}$$

Example: $\mathrm{cDes}(51432) = \{1,3,4\}$, $\mathrm{cDes}(21543) = \{1,3,4,5\}$.

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Introduced by Cellini '95; further studied by Dilks, Petersen and Stembridge '09 among others.

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$$\mathrm{cDes}(\phi(\pi)) = \mathrm{cDes}(\pi) + 1.$$

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(a) $\mathrm{cDes}(\pi) \cap [n-1] = \mathrm{Des}(\pi) \qquad \forall \pi \in \mathcal{S}_n$,

(b) there exists a bijection $\phi : \mathcal{S}_n \to \mathcal{S}_n$ such that

$$\mathrm{cDes}(\phi(\pi)) = \mathrm{cDes}(\pi) + 1.$$

Indeed, we can just define ϕ by

$$\pi_1\pi_2\dots\pi_{n-1}\pi_n \quad \overset{\phi}{\longmapsto} \quad \pi_n\pi_1\pi_2\dots\pi_{n-1}$$

Standard Young Tableaux

A standard Young tableau (SYT) of skew shape λ/μ is a filling of the diagram of λ/μ with the numbers $1, \ldots, n$ (where $n = \#$boxes) so that entries increase along rows and along columns.

Examples:

$\lambda = (4, 3, 1)$

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 8 \\ \hline 3 & 5 & 7 \\ \cline{1-3} 6 \\ \cline{1-1}\end{array}$$

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Examples:

$$\lambda = (4, 3, 1)$$

1	2	4	8
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6			

$$\lambda/\mu = (5, 3, 3, 1)/(2, 1)$$

		2	3	9
	1	5		
4	7	8		
6				

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Denote the set of all SYT of shape λ/μ by $\mathrm{SYT}(\lambda/\mu)$.

Descents of SYT

The descent set of a standard Young tableau T is

$$\mathrm{Des}(T) = \{i : \ i+1 \text{ is in a lower row than } i\}.$$

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Examples:

$$T = \begin{array}{|c|c|c|c|} \hline 1 & \color{red}{2} & \color{red}{4} & 8 \\ \hline 3 & \color{red}{5} & 7 \\ \cline{1-3} 6 \\ \cline{1-1} \end{array} \in \mathrm{SYT}((4,3,1)) \qquad \mathrm{Des}(T) = \{\color{red}{2}, \color{red}{4}, \color{red}{5}\}$$

$$T = \begin{array}{ccccc} \cline{3-5} & & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{\color{red}{3}} & \multicolumn{1}{c|}{9} \\ \cline{2-5} & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{\color{red}{5}} & & \\ \cline{1-3} \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{8} & & \\ \cline{1-3} \multicolumn{1}{|c|}{6} & & & & \\ \cline{1-1} \end{array} \in \mathrm{SYT}((5,3,3,1)/(2,1)) \qquad \mathrm{Des}(T) = \{\color{red}{3}, \color{red}{5}\}$$

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Is there a notion of cyclic descent set on SYT?

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T	$\begin{array}{\|c\|c\|c\|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}$
$\mathrm{Des}(T)$	$\{1,3\}$	$\{2,4\}$	$\{3\}$	$\{1,4\}$	$\{2\}$

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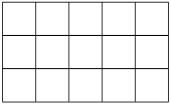
$\mathrm{cDes}(T)$	$\{1,3\}$	$\{2,4\}$	$\{3,5\}$	$\{1,4\}$	$\{2,5\}$

SYT of rectangular shapes



For $r \mid n$, let $\lambda = (r, \dots, r) \vdash n$ be a rectangular shape.

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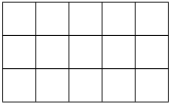

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Theorem (Rhoades '10)

There exists a cyclic descent extension for $\lambda = (r, \ldots, r)$.

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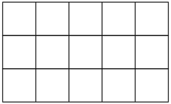

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Here, the bijection ϕ that shifts cDes is Schützenberger's *jeu-de-taquin* promotion operator p.

SYT of rectangular shapes

$$\begin{tikzcd}[column sep=small]
\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array} \arrow[r] \arrow[rrrrr, bend left, "p"] & \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&\ \\\hline\end{array} \arrow[r] & \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&\ &5\\\hline\end{array} \arrow[r] & \begin{array}{|c|c|c|}\hline 1&\ &4\\\hline 2&3&5\\\hline\end{array} \arrow[r] & \begin{array}{|c|c|c|}\hline \ &1&4\\\hline 2&3&5\\\hline\end{array} \arrow[r] & \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array} \arrow[lllll, bend left, "p^{-1}"]
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\end{tikzcd}$$

p determines a $\mathbb{Z}_n$-action. Here are the orbits for $\lambda = (3,3)$:

$$\begin{tikzcd}
T & \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array} \arrow[r, mapsto, "p"] & \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array} \arrow[r, mapsto, "p"] & \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array} \arrow[ll, bend right, "p"'] & \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array} \arrow[r, mapsto, "p"] & \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array} \arrow[l, bend right, "p"'] \\
\mathrm{Des}(T) & \{1,4\} & \{2,5\} & \{3\} & \{1,3,5\} & \{2,4\}
\end{tikzcd}$$

SYT of rectangular shapes

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\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array} \arrow[r] \arrow[rrrrr, bend left, "p"] & \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&\ \\\hline\end{array} \arrow[r] & \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&\ &5\\\hline\end{array} \arrow[r] & \begin{array}{|c|c|c|}\hline 1&\ &4\\\hline 2&3&5\\\hline\end{array} \arrow[r] & \begin{array}{|c|c|c|}\hline \ &1&4\\\hline 2&3&5\\\hline\end{array} \arrow[r] & \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array} \arrow[lllll, bend left, "p^{-1}"]
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\mathrm{Des}(T) & \{1,4\} & & \{2,5\} & & \{3\} & \{1,3,5\} & \{2,4\}
\end{array}$$

To define cDes on $T \in \mathrm{SYT}(r,\ldots,r)$, let

$$n \in \mathrm{cDes}(T) \;\text{ iff }\; n-1 \in \mathrm{Des}(p^{-1}(T)).$$

SYT of rectangular shapes

$$\begin{tikzcd}
\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&{\color{red}5}&6\\\hline\end{array} \arrow[r] \arrow[rrrrr, bend left, "p"] & \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&\ \\\hline\end{array} \arrow[r] & \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&\ &5\\\hline\end{array} \arrow[r] & \begin{array}{|c|c|c|}\hline 1&\ &4\\\hline 2&3&5\\\hline\end{array} \arrow[r] & \begin{array}{|c|c|c|}\hline \ &1&4\\\hline 2&3&5\\\hline\end{array} \arrow[r] & \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&{\color{red}6}\\\hline\end{array} \arrow[lllll, bend left, "p^{-1}"]
\end{tikzcd}$$

${\color{red}5} \notin \mathrm{Des}$ ${\color{red}6} \notin \mathrm{cDes}$

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\mathrm{cDes}(T) & \{1,4\} & \{2,5\} & \{3,{\color{red}6}\} & \{1,3,5\} & \{2,4,{\color{red}6}\}
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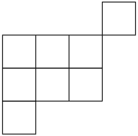
Cyclic descents on SYT($\lambda^{\square}$)

For a partition $\lambda \vdash n-1$, let $\lambda^{\square}$ be the skew shape obtained from λ by placing a disconnected box at its upper right corner.

$$(3,3,1)^{\square} =$$



Cyclic descents on $\mathrm{SYT}(\lambda^{\square})$

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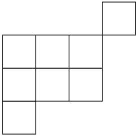
$(3,3,1)^{\square} =$ 

Theorem (E.-Roichman '16)

For every $\lambda \vdash n-1$, there exists a cyclic descent extension for $\lambda^{\square}$.

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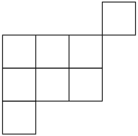
$(3,3,1)^{\square} =$ 

Theorem (E.-Roichman '16)

For every $\lambda \vdash n-1$, there exists a cyclic descent extension for $\lambda^{\square}$.

What is the definition of cDes and ϕ in this case?

Definition of cDes on $\mathrm{SYT}(\lambda^{\square})$

Example:

$$\begin{array}{|c|c|c|}\hline & & 4 \\ \hline 1 & 3 & \\ \hline 2 & & \\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline & & 1 \\ \hline 2 & 4 & \\ \hline 3 & & \\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline & & 2 \\ \hline 1 & 3 & \\ \hline 4 & & \\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline & & 3 \\ \hline 1 & 2 & \\ \hline 4 & & \\ \hline\end{array}$$

$$\{1, {\color{red}4}\} \qquad \{1, 2\} \qquad \{2, 3\} \qquad \{3, {\color{red}4}\}$$

Definition of cDes on $\mathrm{SYT}(\lambda^{\square})$

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$$\{1, 4\} \qquad \{1, 2\} \qquad \{2, 3\} \qquad \{3, 4\}$$

For $T \in \mathrm{SYT}(\lambda^{\square})$, let $n \in \mathrm{cDes}(T)$ iff

- n is strictly higher than 1, or
- $n - d \in \mathrm{Des}(\mathrm{jdt}(T - d))$, where d is the letter in the disconnected cell of T.

Definition of cDes on SYT($\lambda^{\square}$)

Example:

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What is ${\color{red}\mathrm{jdt}(T - d)}$?

A *jeu-de-taquin* straightening algorithm

Given an SYT T with n boxes, let $T+k$ be obtained by adding k mod n to each entry.

$$T = \begin{array}{cccc} & & & 6 \\ 1 & 3 & 5 & \\ 2 & 4 & & \end{array} \qquad T+3 = \begin{array}{cccc} & & & 3 \\ 4 & 6 & 2 & \\ 5 & 1 & & \end{array}$$

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$$T = \begin{array}{|c|c|c|c|} \cline{4-4} \multicolumn{3}{c|}{} & 6 \\ \hline 1 & 3 & 5 & \multicolumn{1}{c}{} \\ \hline 2 & 4 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \qquad T + 3 = \begin{array}{|c|c|c|c|} \cline{4-4} \multicolumn{3}{c|}{} & 3 \\ \hline 4 & 6 & 2 & \multicolumn{1}{c}{} \\ \hline 5 & 1 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$$

Let $\mathrm{jdt}(T + k)$ be the SYT obtained from $T + k$ by repeatedly applying the following step:

- Let i be the minimal entry for which the entry immediately above or to its left is $> i$.
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A *jeu-de-taquin* straightening algorithm

Given an SYT T with n boxes, let $T+k$ be obtained by adding k mod n to each entry.

$$T = \begin{array}{|c|c|c|c|} \hline \multicolumn{3}{c|}{} & 6 \\ \hline 1 & 3 & 5 & \multicolumn{1}{|c}{} \\ \hline 2 & 4 & \multicolumn{2}{|c}{} \\ \cline{1-2} \end{array} \qquad T+3 = \begin{array}{|c|c|c|c|} \hline \multicolumn{3}{c|}{} & 3 \\ \hline 4 & 6 & 2 & \multicolumn{1}{|c}{} \\ \hline 5 & 1 & \multicolumn{2}{|c}{} \\ \cline{1-2} \end{array}$$

Let $\mathrm{jdt}(T+k)$ be the SYT obtained from $T+k$ by repeatedly applying the following step:

- Let i be the minimal entry for which the entry immediately above or to its left is $> i$.
  Switch i with the larger of these two entries.

$$\begin{array}{|c|c|c|c|} \hline \multicolumn{3}{c|}{} & 3 \\ \hline 4 & 6 & 2 & \multicolumn{1}{|c}{} \\ \hline 5 & 1 & \multicolumn{2}{|c}{} \\ \cline{1-2} \end{array} \mapsto \begin{array}{|c|c|c|c|} \hline \multicolumn{3}{c|}{} & 3 \\ \hline 4 & 1 & 2 & \multicolumn{1}{|c}{} \\ \hline 5 & 6 & \multicolumn{2}{|c}{} \\ \cline{1-2} \end{array} \mapsto \begin{array}{|c|c|c|c|} \hline \multicolumn{3}{c|}{} & 3 \\ \hline 1 & 4 & 2 & \multicolumn{1}{|c}{} \\ \hline 5 & 6 & \multicolumn{2}{|c}{} \\ \cline{1-2} \end{array} \mapsto \begin{array}{|c|c|c|c|} \hline \multicolumn{3}{c|}{} & 3 \\ \hline 1 & 2 & 4 & \multicolumn{1}{|c}{} \\ \hline 5 & 6 & \multicolumn{2}{|c}{} \\ \cline{1-2} \end{array} = \mathrm{jdt}(T+3)$$

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Note: promotion is just $p(T) = \mathrm{jdt}(T+1)$, $p^{-1}(T) = \mathrm{jdt}(T-1)$.

Definition of cDes on SYT($\lambda^{\square}$)

$$\begin{array}{ccc} & & \boxed{4} \\ \boxed{1} & \boxed{3} & \\ \boxed{2} & & \end{array} \qquad \begin{array}{ccc} & & \boxed{1} \\ \boxed{2} & \boxed{4} & \\ \boxed{3} & & \end{array} \qquad \begin{array}{ccc} & & \boxed{2} \\ \boxed{1} & \boxed{3} & \\ \boxed{4} & & \end{array} \qquad \begin{array}{ccc} & & \boxed{3} \\ \boxed{1} & \boxed{2} & \\ \boxed{4} & & \end{array}$$

$$\{1,4\} \qquad \{1,2\} \qquad \{2,3\} \qquad \{3,4\}$$

For $T \in \mathrm{SYT}(\lambda^{\square})$, define $n \in \mathrm{cDes}(T)$ iff

- n is strictly north of 1, or
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$$\{1, {\color{red}4}\} \qquad \{1,2\} \qquad \{2,3\} \qquad \{3, {\color{red}4}\}$$

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$4 \in \mathrm{cDes}$ $\qquad$ $4 - 3 = 1 \in \mathrm{Des}$

The bijection ϕ that shifts cDes on $\mathrm{SYT}(\lambda^\square)$

The map $\phi : \mathrm{SYT}(\lambda^\square) \to \mathrm{SYT}(\lambda^\square)$ given by

$$\phi(T) = \mathrm{jdt}\,(\,\mathrm{jdt}(T - d) + d + 1\,)\,,$$

where d is the letter in the disconnected cell of T,
is a bijection such that $\mathrm{cDes}(\phi(T)) = \mathrm{cDes}(T) + 1$ for all T.

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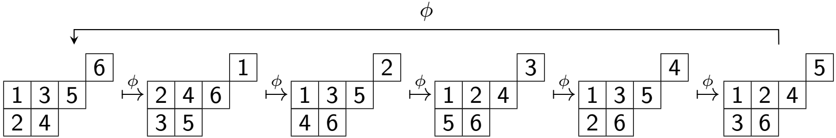
Example:



cDes $\{1,3,6\}$ $\{1,2,4\}$ $\{2,3,5\}$ $\{3,4,6\}$ $\{1,4,5\}$ $\{2,5,6\}$

How about other shapes?

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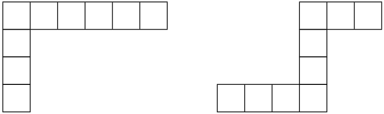
Definition

A connected skew shape λ/μ is a ribbon if it does not contain a 2×2 rectangle.



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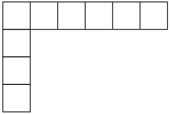

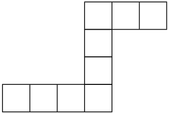

Fact: If λ/μ is a connected ribbon (other than a single row or column), then there is no cyclic descent extension for λ/μ.

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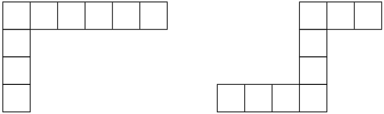

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$$\mathrm{Des}(T) \qquad \{1\} \qquad \{2\} \qquad 3?$$

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Theorem (Adin–Reiner–Roichman '17)

For every skew shape λ/μ that is not a connected ribbon, there is a cyclic descent extension for λ/μ.

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Question: Can we find an explicit description of cDes for other shapes λ/μ?

Explicit description of cDes for some shapes

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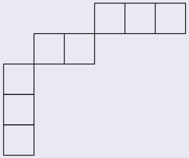

(*strip*)

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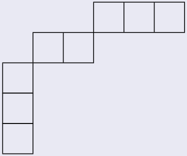

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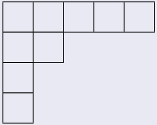

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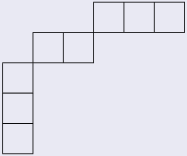

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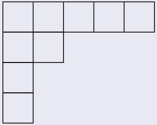

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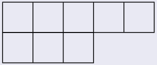

(*two-row straight*)

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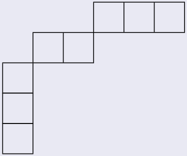

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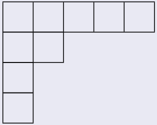

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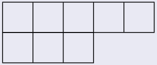

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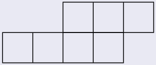
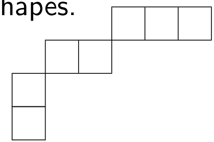


(*two-row skew*)

Definition of cDes on strips

Let λ/μ be a strip of size n, i.e., a shape whose components are one-row or one-column shapes.



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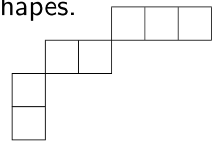

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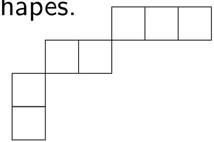

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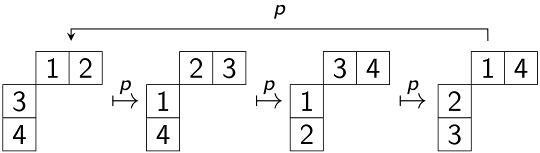
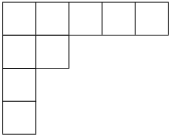
Again, the promotion operator $p : T \mapsto \mathrm{jdt}(T+1)$ shifts cDes:



cDes $\{2,3\}$ $\{3,4\}$ $\{1,4\}$ $\{1,2\}$

cDes on hooks plus a box

Let $\lambda = (n-k-2, 2, 1^k)$, where $0 \leq k \leq n-4$.



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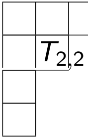

For $T \in \mathrm{SYT}(\lambda)$, let $n \in \mathrm{cDes}(T)$ iff

- $T_{2,2} - 1$ is in the first column of T.

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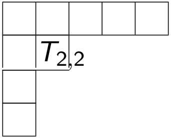

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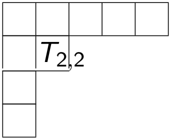

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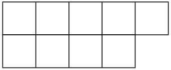
We have a complicated explicit definition of a bijection ϕ that shifts cDes. In this case it doesn't determine a $\mathbb{Z}_n$-action.

cDes on two-row straight shapes

Let $\lambda = (n-k, k)$, where $2 \leq k \leq n/2$.



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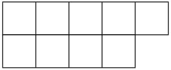

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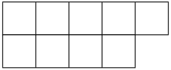

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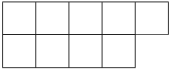

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Examples:

$$9 \in \mathrm{cDes}\left(\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & 9 \\ \hline 4 & 6 & 7 & 8 & \\ \cline{1-4}\end{array}\right) \text{ because } 8 = 7 + 1,\ 4 > 2 \text{ and } 6 > 3.$$

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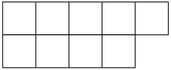

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$9 \in \mathrm{cDes}\left(\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & 9 \\ \hline 4 & 6 & 7 & 8 \\ \cline{1-4}\end{array}\right)$ because $8 = 7 + 1$, $4 > 2$ and $6 > 3$.

$9 \notin \mathrm{cDes}\left(\begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 6 & 9 \\ \hline 2 & 5 & 7 & 8 \\ \cline{1-4}\end{array}\right)$ because $2 < 3$.

cDes on two-row straight shapes

- When $\lambda = (n-2, 2)$, the definition of cDes viewed as a two-row shape coincides with the definition viewed as a hook plus a box.

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- For $\lambda = (r, r)$, the definition of cDes viewed as a two-row shape coincides with Rhoades' definition viewed as a rectangular shape.

ϕ on two-row straight shapes

For two-row straight shapes, we have an explicit definition of a map ϕ that shifts cDes, but it does not determine a $\mathbb{Z}_n$-action.

$$\begin{array}{|c|c|c|c|c|}\hline \textcolor{red}{1}&\textcolor{red}{3}&5&6&\textcolor{red}{7}\\ \hline 2&4&8&9\\ \cline{1-4}\end{array} \overset{\phi}{\mapsto} \begin{array}{|c|c|c|c|c|}\hline 1&\textcolor{red}{2}&\textcolor{red}{4}&7&\textcolor{red}{8}\\ \hline 3&5&6&9\\ \cline{1-4}\end{array} \overset{\phi}{\mapsto} \begin{array}{|c|c|c|c|c|}\hline 1&2&\textcolor{red}{3}&\textcolor{red}{5}&\textcolor{red}{9}\\ \hline 4&6&7&8\\ \cline{1-4}\end{array} \overset{\phi}{\mapsto} \begin{array}{|c|c|c|c|c|}\hline \textcolor{red}{1}&3&\textcolor{red}{4}&\textcolor{red}{6}&9\\ \hline 2&5&7&8\\ \cline{1-4}\end{array} \overset{\phi}{\mapsto} \begin{array}{|c|c|c|c|c|}\hline 1&\textcolor{red}{2}&\textcolor{red}{5}&\textcolor{red}{7}&9\\ \hline 3&4&6&8\\ \cline{1-4}\end{array}$$

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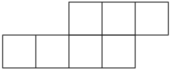
(cDes in red)

cDes on two-row skew shapes

Let $\lambda/\mu = (n-k+m, k)/(m)$ with $k \neq m+1$.



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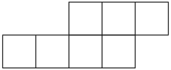

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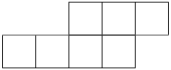

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We have no explicit description of ϕ in this case.

Non-uniqueness of cDes

For many shapes, the definition of cDes is not unique.

Example: Let $\lambda/\mu = (4,2)/(2)$.

$$\begin{array}{cccc} & & 1 & 4 \\ 2 & 3 & & \end{array} \quad \begin{array}{cccc} & & 1 & 2 \\ 3 & 4 & & \end{array} \quad \begin{array}{cccc} & & 2 & 3 \\ 1 & 4 & & \end{array} \quad \begin{array}{cccc} & & 3 & 4 \\ 1 & 2 & & \end{array} \quad \begin{array}{cccc} & & 1 & 3 \\ 2 & 4 & & \end{array} \quad \begin{array}{cccc} & & 2 & 4 \\ 1 & 3 & & \end{array}$$

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Our definition of cDes:

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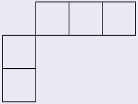
Another possible definition of cDes:

$$\{1\} \qquad \{2,4\} \qquad \{3\} \qquad \{4\} \qquad \{1,3\} \qquad \{2\}$$

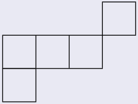
Uniqueness of cDes for near-hooks

Theorem (Adin-E.-Roichman '17)

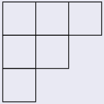
Suppose that either λ/μ or its 180^o-rotation is "one cell away from a hook", i.e.



hook minus its corner cell



hook plus a disconnected cell



hook plus an internal cell

Then cDes *on* SYT(λ/μ) *is uniquely defined.*

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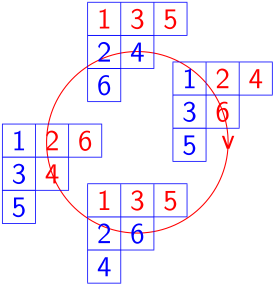

vs.

(cDes in red)

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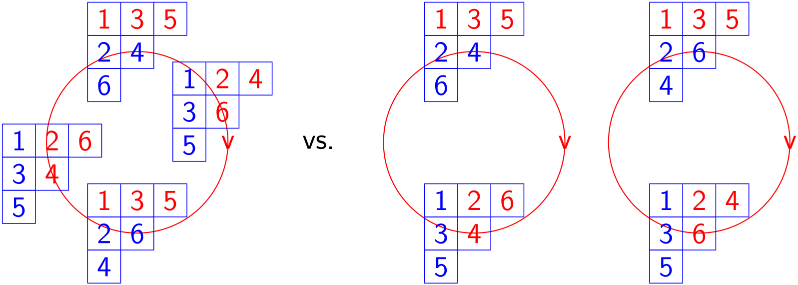

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Open problems

For each non-ribbon shape λ/μ:

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  (Adin–Reiner–Roichman prove that such an involution exists.)

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Thanks!

Also:

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Dartmouth College
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