Consecutive Patterns in Inversion Sequences

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Joint work with Juan Auli

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Special Session on Patterns in Permutations

Inversion sequences

An inversion sequence of length n is an integer sequence $e = e_1 e_2 \cdots e_n$ such that $0 \leq e_i < i$.

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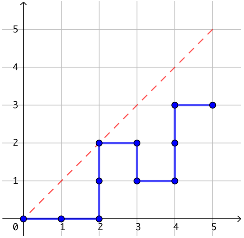
Example. $e = 00213 \in \mathbf{I}_5$.



Permutations can be encoded as inversion sequences via the bijection $\Theta : S_n \to \mathbf{I}_n$, defined by $\Theta(\pi) = e_1 e_2 \cdots e_n$ where

$$e_i = |\{j : j < i \text{ and } \pi_j > \pi_i\}|.$$

For instance, $\Theta(35142) = 00213$.

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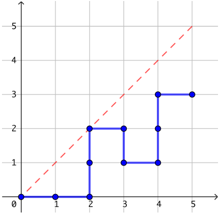

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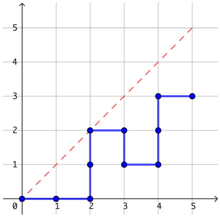


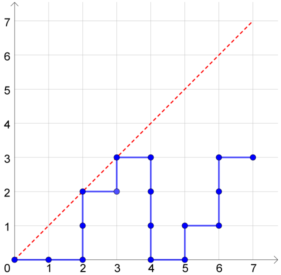
Let $\mathbf{I}_n(p) = \{e \in \mathbf{I}_n : e \text{ avoids } p\}$.
For example, $\mathbf{I}_3(001) = \{000, 010, 011, 012\}$.

The avoidance sequences $|\mathbf{I}_n(p)|$ have been studied by Corteel–Martinez–Savage–Weselcouch and by Mansour–Shattuck.
Go to Megan's talk tomorrow to hear more about this!

Consecutive patterns in inversion sequences

$e \in \mathbf{I}_n$ **contains** the (consecutive) pattern $p = \underline{p_1 p_2 \cdots p_l}$ if there is a consecutive subsequence $e_i e_{i+1} \cdots e_{i+l-1}$ whose reduction is p. Otherwise, e **avoids** p.

Example. $e = 0023013$ contains $\underline{012}$ and $\underline{120}$, but it avoids $\underline{000}$ and $\underline{010}$.



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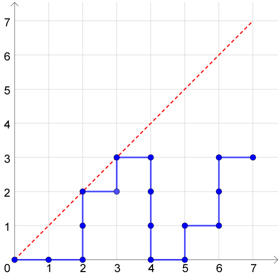

$\mathbf{I}_n(p) = \{e \in \mathbf{I}_n : e \text{ avoids } p\}$.

Goal 1: determine $|\mathbf{I}_n(p)|$ for consecutive patterns $p = \underline{p_1 p_2 \cdots p_l}$.

Avoiding consecutive patterns of length 3

Let $\mathbf{I}_{n,k}(p) = \{e \in \mathbf{I}_n(p) : e_n = k\}$, so that $\mathbf{I}_n(p) = \bigcup_{k=0}^{n-1} \mathbf{I}_{n,k}(p)$.

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Pattern p	$\lvert\mathbf{I}_n(p)\rvert$ in the OEIS	Recurrence for $\lvert\mathbf{I}_{n,k}(p)\rvert$
$\underline{012}$	A049774*, equals $\lvert S_n(\underline{321})\rvert$	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - \sum_{l=1}^{k-1}\sum_{j=0}^{l-1}\sum_{i\geq j}\lvert\mathbf{I}_{n-3,i}(p)\rvert$
$\underline{021}$	A071075*, equals $\lvert S_n(\underline{132}4)\rvert$	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - (n-2-k)\sum_{j=0}^{k-1}\lvert\mathbf{I}_{n-2,j}(p)\rvert$
$\underline{102}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - \sum_{j\geq 1} j\,\lvert\mathbf{I}_{n-2,j}(p)\rvert$
$\underline{120}$	A200404, equals $\lvert S_n(1\underline{432})\rvert$	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - \sum_{j>k}(n-2-j)\,\lvert\mathbf{I}_{n-2,j}(p)\rvert$
$\underline{201}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - k\sum_{j>k}\lvert\mathbf{I}_{n-2,j}(p)\rvert$
$\underline{210}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - \sum_{l=k+1}^{n-4}\sum_{j=l+1}^{n-3}\sum_{i\leq j}\lvert\mathbf{I}_{n-3,i}(p)\rvert$

* Formulas were known for these sequences.

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$\underline{000}$	A052169*	$\lvert\mathbf{I}_n(p)\rvert = \frac{(n+1)!-d_{n+1}}{n}$, where $d_n = \#$ derangements
$\underline{001}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - \sum_{j<k}\lvert\mathbf{I}_{n-2,j}(p)\rvert$
$\underline{010}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - (n-2-k)\,\lvert\mathbf{I}_{n-2,k}(p)\rvert$
$\underline{011}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - \sum_{j<k}\lvert\mathbf{I}_{n-2,j}(p)\rvert$ (if $k \neq n-1$)
$\underline{100}, \underline{110}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - \sum_{j>k}\lvert\mathbf{I}_{n-2,j}(p)\rvert$
$\underline{101}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - k\,\lvert\mathbf{I}_{n-2,k}(p)\rvert$

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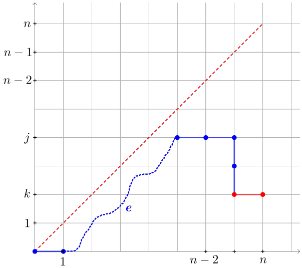
For $p = \underline{110}$:

$$|\mathbf{I}_{n,k}(\underline{110})| = |\mathbf{I}_{n-1}(\underline{110})| - \sum_{j>k} |\mathbf{I}_{n-2,j}(\underline{110})|.$$



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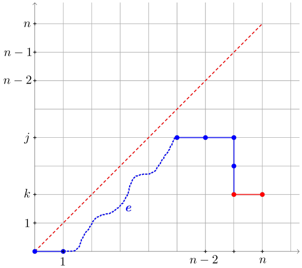

For $p = \underline{000}$:

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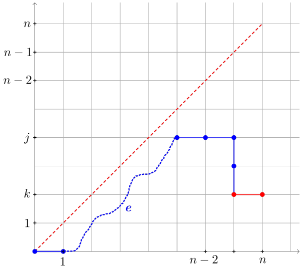

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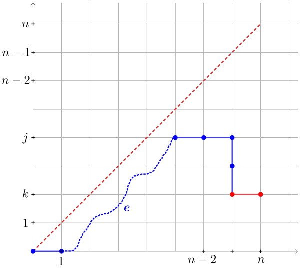

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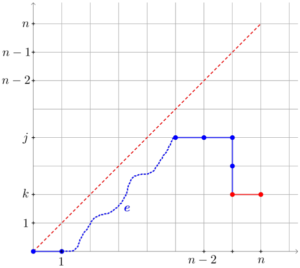

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More generally, for $p = \underline{0^r}$:

$$|\mathbf{I}_n(\underline{0^r})| = \sum_{j=1}^{r-1} (n-j)\,|\mathbf{I}_{n-j}(\underline{0^r})|.$$

Equivalences between patterns

For $e \in \mathbf{I}_n$ and a consecutive pattern p, let

$$\mathrm{Oc}(p, e) = \{i : e_i e_{i+1} e_{i+2} \text{ is an occurence of } p\}.$$

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Goal 2: classify consecutive patterns into these equivalence classes.

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Proof sketch.

1. For any $S \subseteq [n]$, construct a bijection

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This is the only equivalence between consecutive patterns of length 3.

Patterns of length 4

Theorem. A complete list of equivalences between consecutive patterns of length 4 is as follows:

- $\underline{0102} \overset{ss}{\sim} \underline{0112}$
- $\underline{0021} \overset{ss}{\sim} \underline{0121}$
- $\underline{1002} \overset{ss}{\sim} \underline{1012} \overset{ss}{\sim} \underline{1102}$
- $\underline{0100} \overset{ss}{\sim} \underline{0110}$
- $\underline{2013} \overset{ss}{\sim} \underline{2103}$
- $\underline{1200} \overset{ss}{\sim} \underline{1210} \overset{ss}{\sim} \underline{1220}$
- $\underline{0211} \overset{ss}{\sim} \underline{0221}$
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- $\underline{2012} \overset{ss}{\sim} \underline{2102}$
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Conjecture. If p and p' are consecutive patterns of length m in inversion sequences, then

$$p \sim p' \iff p \overset{s}{\sim} p'$$

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Conjecture. If p and p' are consecutive patterns of length m in inversion sequences, then

$$p \sim p' \iff p \overset{s}{\sim} p'$$

Analogous to Nakamura's conjecture for consecutive patterns in permutations.

Patterns of length 4

Theorem. A complete list of equivalences between consecutive patterns of length 4 is as follows:

- $\underline{0102} \overset{ss}{\sim} \underline{0112}$
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Proof ideas

Equivalences are proved differently, but there are three main cases:

- p and p' are non-overlapping, mutually non-overlapping and "interchangeable". **Example:** $\underline{1002} \stackrel{ss}{\sim} \underline{1012}$.

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  Proof is bijective, and distribution of occurrences is symmetric:

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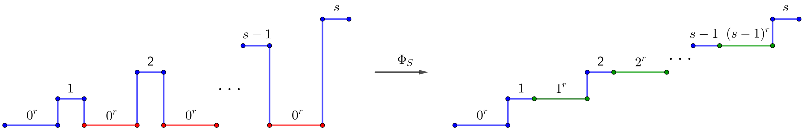
The 75 consecutive patterns of length 4 fall into 55 equivalence classes.

Longer patterns

Some equivalences generalize to longer patterns:

Theorem. For every $r \geq 1$ and $s \geq 2$,

$$\underline{0^r\,1\,0^r\,2\,0^r\ldots(s-1)\,0^r\,s} \overset{ss}{\sim} \underline{0^r\,1\,1^r\,2\,2^r\ldots(s-1)\,(s-1)^r\,s}$$



$$\underline{s\,0^r\,(s-1)\,0^r\ldots0^r\,10^r} \overset{ss}{\sim} \underline{s\,(s-1)^r\,s\,(s-2)^r\,s\ldots s\,1^r s\,0^r}$$

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Patterns of relations

Let $R_1, R_2 \in \{\leq, \geq, <, >, =, \neq\}$.

$e \in \mathbf{I}_n$ contains the (consecutive) pattern of relations $(\underline{R_1, R_2})$ if there is an i such that $e_i R_1 e_{i+1}$ and $e_{i+1} R_2 e_{i+2}$.
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We define the relations $\sim$, $\overset{s}{\sim}$ and $\overset{ss}{\sim}$ for patterns of relations like we did for patterns.

Goal 3: Classify patterns of relations into equivalence classes and determine $|\mathbf{I}_n(\underline{R_1, R_2})|$.

Equivalences between patterns of relations

Theorem. A complete list of equivalences between consecutive patterns of relations $(\underline{R_1, R_2})$ is as follows:

- $(\underline{\geq, <}) \stackrel{ss}{\sim} (\underline{<, \geq}) \sim (\underline{\neq, \geq})$
- $(\underline{\geq, \geq}) \stackrel{ss}{\sim} (\underline{<, <})$
- $(\underline{\geq, =}) \stackrel{ss}{\sim} (\underline{=, \geq})$
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Corollary (conjectured by Baxter–Pudwell, proved non-bijectively by Baxter–Shattuck and Kasraoui)**.** The vincular permutation patterns $\underline{124}3$ and $\underline{421}3$ are Wilf equivalent.

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New bijective proof:
$S_n(\underline{124}3) \leftrightarrow \mathbf{I}_n(\underline{>, \geq}) \leftrightarrow \mathbf{I}_n(\underline{\geq, >}) \leftrightarrow S_n(\underline{421}3)$.

Avoiding patterns of relations

Patterns of relations for which the sequence $|\mathbf{I}_n(\underline{R_1, R_2})|$ appears in the OEIS as enumerating other objects:

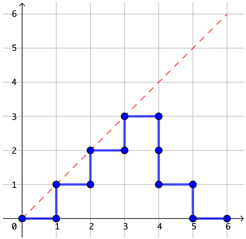
Pattern $(\underline{R_1, R_2})$	OEIS	Description
$(\underline{\leq, \neq})$	A040000	2 (for $n > 1$)
$(\underline{\leq, \geq})$	A000027	n
$(\underline{\geq, \neq})$	A000124	$\binom{n}{2} + 1$
$(\underline{\geq, \leq})$	A000045	F_{n+1} (Fibonacci)
$(\underline{\neq, \leq})$	A000071	$F_{n+2} - 1$ (Fibonacci)
$(\underline{\geq, <}) \overset{ss}{\sim} (\underline{<, \geq}) \sim (\underline{\neq, \geq})$	A000079	2^{n-1}
$(\underline{\neq, \neq})$	A000085	Number of involutions of $[n]$
$(\underline{\leq, >})$	A000108	C_n (Catalan)
$(\underline{>, \leq})$	A071356	Underdiagonal paths of from the origin to $x = n$ with steps $(0,1)$, $(1,0)$, $(1,2)$
$(\underline{=, \neq})$	A003422	$0! + 1! + 2! + \cdots + (n-1)!$
$(\underline{\geq, \geq}) \overset{ss}{\sim} (\underline{<, <})$	A049774	$\|S_n(\underline{321})\|$
$(\underline{\neq, =})$	A000522	$\sum_{i=0}^{n-1} (n-1)!/i!$
$(\underline{\geq, >}) \overset{ss}{\sim} (\underline{>, \geq})$	A200403	$\|S_n(\underline{1243})\|$
$(\underline{=, =})$	A052169	$\frac{(n+1)! - d_{n+1}}{n}$

Examples

Let $e \in \mathbf{I}_n$.

$e \in \mathbf{I}_n(\underline{\geq, \leq})$ iff there exists j such that

$e_1 < e_2 < \cdots < e_j \geq e_{j+1} > e_{j+2} > \cdots > e_n$.



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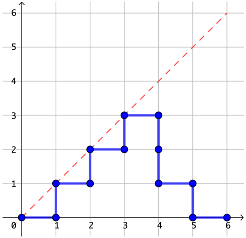
$\Longrightarrow \quad |\mathbf{I}_n(\underline{\geq, \leq})| = F_{n+1}$.



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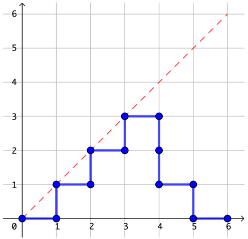
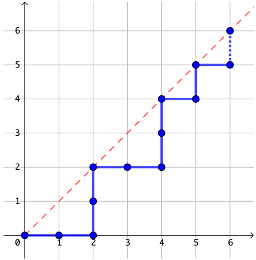


$e \in \mathbf{I}_n(\underline{\leq, >})$ iff $e_1 \leq e_2 \leq \cdots \leq e_n$.



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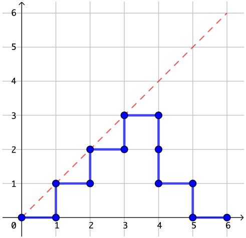

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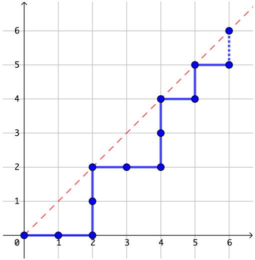

The pattern $\mathbf{I}_n(\underline{>, \leq})$

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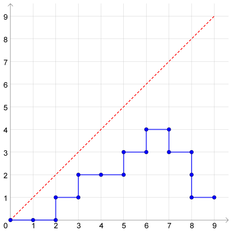

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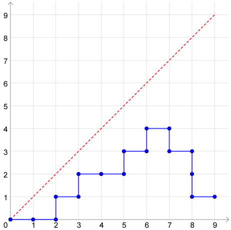

Theorem (conjectured by Martinez–Savage, proved independently by Cao–Jin–Lin and Hossain).

$$\sum_{n\geq 0} \left|\mathbf{I}_n(\underline{\geq, \leq})\right| x^n = \frac{1-2x-\sqrt{1-4x-4x^2}}{4x^2}.$$

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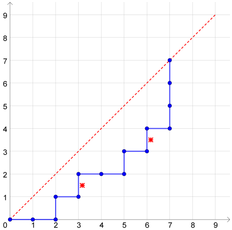

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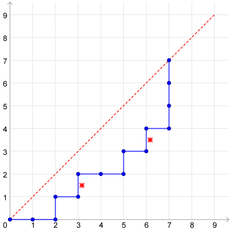

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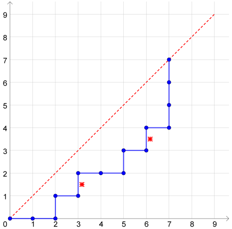

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- enumeration formulas for inversion sequences satisfying other unimodality conditions.

References

- Juan S. Auli and Sergi Elizalde, *Consecutive patterns in inversion sequences*, Discrete Math. Theor. Comput. Sci. 21 (2019), #6.
- Juan S. Auli and Sergi Elizalde, *Consecutive patterns in inversion sequences II: avoiding patterns of relations*, J. Integer Seq. 22 (2019), Art. 19.7.5.
- Andrew M. Baxter and Lara K. Pudwell, *Enumeration schemes for vincular patterns*, Discrete Math., **312** (2012), 1699–1712.
- Andrew Baxter and Mark Shattuck, *Some Wilf-equivalences for vincular patterns*, J. Comb., **6** (2015), 19–45.
- Wenqin Cao, Emma Yu Jin and Zhicong Lin, *Enumeration of inversion sequences avoiding triples of relations*, Discrete Appl. Math., **260** (2019), 86–97.
- Sylvie Corteel, Megan A. Martinez, Carla D. Savage and Michael Weselcouch, *Patterns in inversion sequences I*, Discrete Math. Theor. Comput. Sci., **18** (2016), 21 pp.
- Tim Dwyer and Sergi Elizalde, *Wilf equivalence relations for consecutive patterns*, Adv. in Appl. Math., **99** (2018), 134–157.
- Anisse Kasraoui, *New Wilf-equivalence results for vincular patterns*, European J. Combin., **34** (2013), 322–337.
- Toufik Mansour and Mark Shattuck, *Pattern avoidance in inversion sequences*, Pure Math. Appl. (PU.M.A.), **25** (2015),157–176.
- Megan A. Martinez and Carla D. Savage, *Patterns in inversion sequences II: Inversion sequences avoiding triples of relations*, J. Integer Seq., **21** (2018), Art. 18.2.2.