Maximal Brill–Noether Loci via K3 Surfaces

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Abstract

We explain a strategy for distinguishing Brill-Noether loci in the moduli space of curves by studying the lifting of linear systems on curves in polarized K3 surfaces, which motivates a conjecture identifying the maximal Brill-Noether loci with respect to containment. Via an analysis of the stability of Lazarsfeld-Mukai bundles, we obtain new lifting results for linear systems of rank 3 which suffice to prove the maximal Brill-Noether loci conjecture in genus 9-19, 22, and 23.

Brill–Noether Loci

The Brill–Noether theorem states that when $\rho(g,r,d) = g-(r+1)(g-d+r) \geq 0$, then every curve of genus g admits a line bundle of type g^r_d. When $\rho(g,r,d) < 0$, the *Brill–Noether locus* $\mathcal{M}^r_{g,d}$ is a proper subvariety of $\mathcal{M}_g$.

There are many containments among Brill–Noether loci [7] :

- $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}^r_{g,d+1}$ when $\rho(g,r,d+1) < 0$, and
- $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}^{r-1}_{g,d-1}$ when $r \geq 2$ and $\rho(g,r-1,d-1) < 0$.

The *expected maximal Brill–Noether loci* are the $\mathcal{M}^r_{g,d}$, where for each r, d is maximal such that $\rho(g,r,d) < 0$ and $\rho(g,r-1,d-1) \geq 0$.

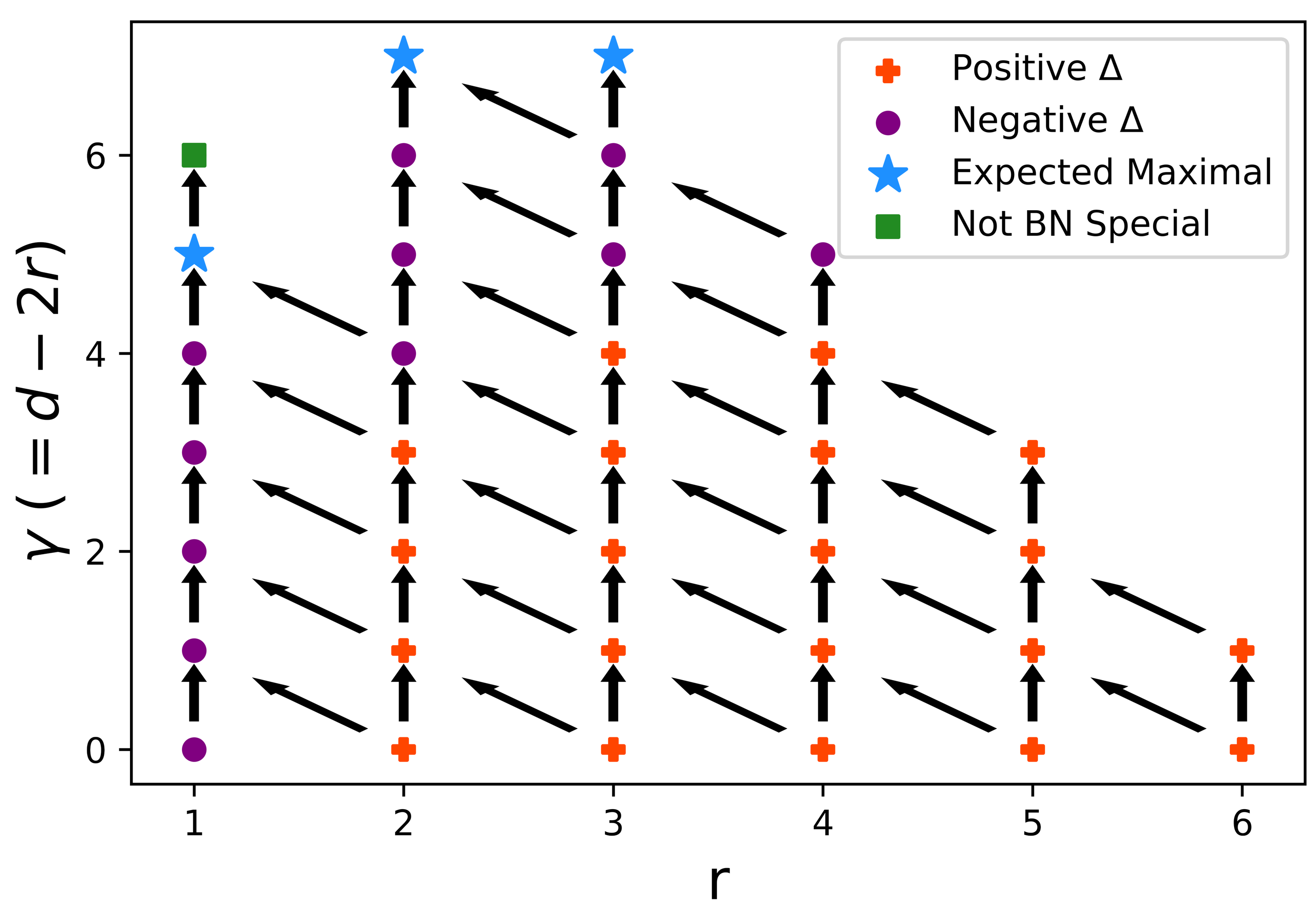

Figure 1. g^r_ds in genus 14. Arrows show containments of the corresponding Brill–Noether loci. The general Clifford index (γ) is 6. $\Delta(g,r,d) = 4(g-1)(r-1) - d^2$.

Conjecture and Theorem

- **Maximal Brill–Noether Locus Conjecture:** In genus $g \geq 9$, the maximal Brill–Noether loci are the expected maximal ones.
- **Theorem:** The conjecture holds in genus $9-19$, 22, and 23.

In genus 20, 21, and $g \geq 24$, we cannot show that some of the expected maximal Brill–Noether loci are not contained in the expected maximal $\mathcal{M}^4_{g,d}$. If we knew that $\operatorname{codim} \mathcal{M}^r_{g,d} = -\rho(g,r,d)$ for $\rho = -4$ and $\rho = -5$ in these cases, then the conjecture holds in genus 20 and 21.

In genus 23, the Brill–Noether loci with $\rho = -1$ were proven to be maximal by Eisenbud–Harris and Farkas. Namely, the divisors $\mathcal{M}^1_{23,12}$, $\mathcal{M}^2_{23,17}$, and $\mathcal{M}^3_{23,20}$ have distinct support in $\mathcal{M}_{23}$. [2, 3, 4]

K3 Surfaces

The Noether–Lefschetz divisor $\mathcal{K}^r_{g,d}$ is the locus of polarized K3 surfaces (S,H) of genus g such that

$$\Lambda^r_{g,d} = \begin{array}{c|cc} & H & L \\ \hline H & 2g-2 & d \\ L & d & 2r-2 \end{array}$$

admits a primitive embedding in $\mathrm{Pic}(S)$ preserving H.

Proposition: Let $(S,H) \in \mathcal{K}^r_{g,d}$ and let $C \in |H|$ be a smooth irreducible curve. If L and $H-L$ are basepoint free, $r \geq 2$, and $0 < d \leq g-1$, then $L \otimes \mathcal{O}_C$ is a g^r_d.

Distinguishing Brill–Noether Loci and Lifting g^r_ds

Our strategy is to show that a curve on a very general polarized K3 surface in $\mathcal{K}^r_{g,d}$ admits a g^r_d, but no other expected maximal $g^{r'}_{d'}$. We do this by studying the lifting of line bundles on polarized K3 surfaces. [4, 5]

Donagi–Morrison Conjecture [1, 6]: Let (S,H) be a polarized K3 surface and $C \in |H|$ be a smooth irreducible curve of genus ≥ 2. Suppose A is a complete basepoint free g^r_d on C such that $d \leq g-1$ and $\rho(g,r,d) < 0$. Then there exists a line bundle $M \in \mathrm{Pic}(S)$ adapted to $|H|$ such that

- $|A|$ is contained in the restriction of $|M|$ to C, and
- $\gamma(M \otimes \mathcal{O}_C) \leq \gamma(A)$.

Donagi and Morrison verified the conjecture for $r=1$, and Lelli-Chiesa verified it for $r=2$ [1, 5], she also verified it under a technical hypothesis that the pair (C,A) do not have any *unexpected secant varieties up to deformation* [6].

Distinguishing Lattices If we have a lifting result, we find conditions on the Picard lattice associated to maximal Brill–Noether loci that would imply the Maximal Brill–Noether conjecture.

(L2): For a fixed $\Lambda^r_{g,d}$ associated to an expected maximal $\mathcal{M}^r_{g,d}$ and any $\Lambda^{r'}_{g,d'}$ with $\lfloor \frac{g+1}{2} \rfloor \leq \gamma(r',d') \leq \lfloor g - 2\sqrt{g} + 1 \rfloor$, and $1 \leq r' \leq \lfloor \frac{g-1-\gamma(r',d')}{2} \rfloor$, one has $\Lambda^{r'}_{g,d'} \nsubseteq \Lambda^r_{g,d}$.

Proposition If the Donagi–Morrison conjecture and L2 hold for all expected maximal g^r_d in genus g, then the Maximal Brill–Noether locus conjecture holds in genus g.

The genera ≤ 200 where **L2** does not hold are genus $89, 91, 92, 145, 153$, and 190. And thus in all other genera below 200 the Donagi–Morrison conjecture implies the Maximal Brill–Noether conjecture.

Genus 14

The expected maximal Brill–Noether loci are $\mathcal{M}^1_{14,8}$, $\mathcal{M}^2_{14,11}$, and $\mathcal{M}^3_{14,13}$. Work of Lelli-Chiesa shows that $\mathcal{M}^3_{14,13} \nsubseteq \mathcal{M}^2_{14,11}$. Recent work on Brill–Noether theory for curves of fixed gonality shows that $\mathcal{M}^1_{14,8}$ is maximal. Moreover, using Lelli-Chiesa's lifting results, it can be shown that $\mathcal{M}^2_{14,11}, \mathcal{M}^3_{14,13} \nsubseteq \mathcal{M}^1_{14,8}$. It remains to find a curve with a g^2_{11} that does not admit a g^3_{13}.

Lifting g^3_ds

Theorem: Let (S,H) be a polarized K3 surface of genus $g \neq 2,3,4,8$, and $C \in |H|$ a smooth irreducible curve of Clifford index $\gamma(C)$. Suppose that S has no elliptic curves and $d < \frac{5}{4}\gamma(C) + 6$, then the Donagi–Morrison conjecture holds for any g^3_d on C.

We prove a slightly more refined version, replacing the hypothesis on non-existence of elliptic curves with an explicit dependence on the Picard lattice of S.

Proof Idea

Let A be a line bundle of type g^3_d on $C \in |H|$. If $\rho(g,r,d) < 0$, then $E_{C,A}$ is not stable. To obtain a Donagi–Morrison lift of A, we want to show that $E_{C,A}$ has a maximal destabilizing subline bundle. To do this, we find lower bounds on d whenever $E_{C,A}$ has a different destabilizing subsheaf by analyzing the Harder–Narasimhan and Jordan–Hölder filtrations.

Lazarsfeld–Mukai Bundles

We define a bundle $F_{C,A}$ on S via the short exact sequence

$$0 \longrightarrow F_{C,A} \longrightarrow H^0(C,A) \otimes \mathcal{O}_S \xrightarrow{ev} \iota_*(A) \longrightarrow 0.$$

Dualizing gives $E_{C,A} = F^\vee_{C,A}$ (the LM bundle associated to A on C) sitting in the short exact sequence

$$0 \longrightarrow H^0(C,A)^\vee \otimes \mathcal{O}_S \longrightarrow E_{C,A} \longrightarrow \iota_*(\omega_C \otimes A^\vee) \longrightarrow 0;$$

The LM bundle $E_{C,A}$ is like a lift of A to a vector bundle on S.

Let $E_{C,A}$ be a LM bundle associated to a basepoint free line bundle A of type g^r_d on $C \subset S$, then:

- $c_1(E_{C,A}) = [C]$ and $c_2(E_{C,A}) = \deg(A)$;
- $\mathrm{rk}(E_{C,A}) = r+1$ and $E_{C,A}$ is globally generated off the base locus of $\iota_*(\omega_C \otimes A^\vee)$;
- $h^0(S,E_{C,A}) = h^0(C,A) + h^0(C,\omega_C \otimes A^\vee) = 2r+1+g-d = g-(d-2r)+1$;
- $h^1(S,E_{C,A}) = h^2(S,E_{C,A}) = 0$, $h^0(S,E^\vee_{C,A}) = h^1(S,E^\vee_{C,A}) = 0$;
- $\chi(F_{C,A} \otimes E_{C,A}) = 2(1-\rho(g,r,d))$.

LM bundles are useful for lifting g^r_ds. In fact, if there is a nontrivial $N \in \mathrm{Pic}(S)$ with $h^0(S,N) \neq 0$, $h^1(S,N) = 0$, and an injection $N \hookrightarrow E_{C,A}$, then the Donagi–Morrison conjecture holds with $L = \mathcal{O}_S(C) \otimes N^\vee$. [6]

Stability of Sheaves on K3 Surfaces

The *slope* of E is $\mu(E) = \frac{c_1(E).H}{\mathrm{rk}(E)}$. A torsion-free coherent sheaf is called slope stable or μ-stable (μ-semistable) if $\mu(F) < \mu(E)$ (respectively, $\mu(F) \leq \mu(E)$) for all coherent sheaves $F \subseteq E$ with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$.

Every torsion-free coherent sheaf E has a unique Harder–Narasimhan filtration, which is an increasing filtration

$$0 = HN_0(E) \subset HN_1(E) \subset \cdots \subset HN_\ell(E) = E,$$

such that the factors $gr^{HN}_i(E) = HN_i(E)/HN_{i-1}(E)$ for $1 \leq i \leq \ell$ are torsion free semistable sheaves with $\mu(gr^{HN}_1(E)) > \mu(gr^{HN}_2(E)) > \cdots > \mu(gr^{HN}_\ell(E))$.

Likewise, every semistable sheaf E has a Jordan–Hölder filtration, which is an increasing filtration with stable factors all of slope $\mu(E)$.

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