Maximal Brill–Noether Loci via K3 Surfaces

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Abstract

The Brill–Noether (BN) loci $\mathcal{M}^r_{g,d}$ parameterize curves of genus g admitting a line bundle of degree d and $r+1$ global sections, and when the BN number is negative, sit as proper subvarieties of the moduli space of genus g curves. We explain a strategy for distinguishing BN loci by studying the lifting of line bundles on polarized K3 surfaces, which motivates a conjecture identifying the maximal Brill–Noether loci. Via an analysis of the stability of Lazarsfeld–Mukai bundles, we obtain new lifting results for line bundles of type g^3_d which suffice to prove the maximal BN loci conjecture in genus $9-19$, 22, and 23.

Brill–Noether Loci

The Brill–Noether theorem states that when $\rho(g,r,d)=g-(r+1)(g-d+r)\geq 0$, then every curve of genus g admits a line bundle of type g^r_d. When $\rho(g,r,d)<0$, the *Brill–Noether locus* $\mathcal{M}^r_{g,d}$ is a proper subvariety of $\mathcal{M}_g$.

There are many containments among BN loci:

- $\mathcal{M}^r_{g,d}\subseteq\mathcal{M}^r_{g,d+1}$, and
- $\mathcal{M}^r_{g,d}\subseteq\mathcal{M}^{r-1}_{g,d-1}$ when $\rho(g,r-1,d-1)<0$.

The *expected maximal Brill–Noether loci* are the $\mathcal{M}^r_{g,d}$, where for each r, d is maximal such that $\rho(g,r,d)<0$ and $\rho(g,r-1,d-1)\geq 0$.

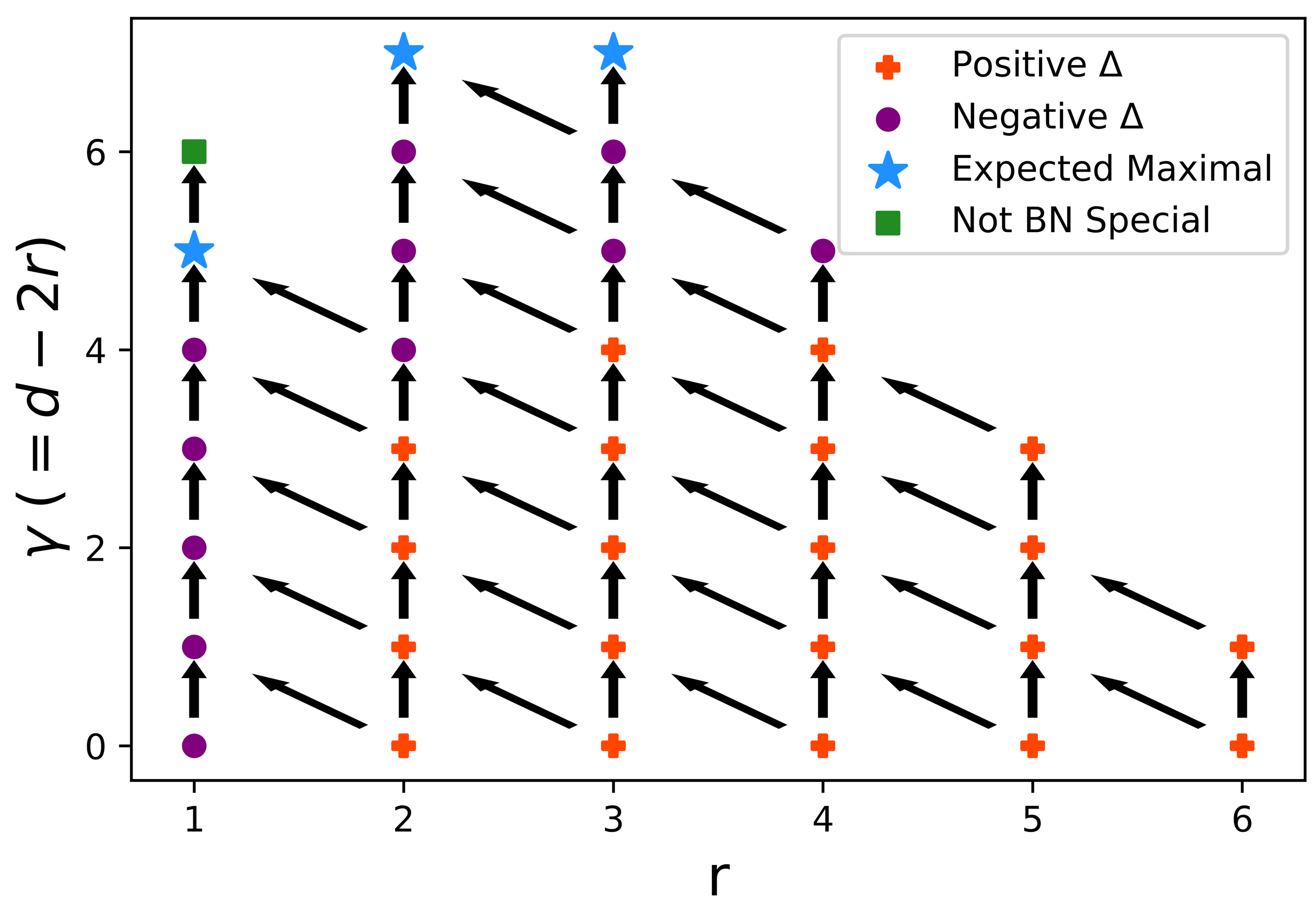

Figure 1. g^r_ds in genus 14. Arrows show containments of the corresponding Brill–Noether loci. The general Clifford index (γ) is 6.

Conjecture and Theorem

- **Conjecture*:** In genus $g\geq 9$, the maximal Brill–Noether loci are the expected maximal ones.
- **Theorem:** The conjecture holds in genus $9-19$, 22, and 23.

In genus 20, 21, and $g\geq 24$, we cannot show that some of the expected maximal Brill–Noether loci are not contained in the expected maximal $\mathcal{M}^4_{g,d}$. If we knew that $\operatorname{codim}\mathcal{M}^r_{g,d}=-\rho(g,r,d)$ for $\rho=-4$ and $\rho=-5$ in these cases, then the conjecture holds in genus 20 and 21.

K3 Surfaces

The Noether–Lefschetz divisor $\mathcal{K}^r_{g,d}$ is the locus of polarized K3 surfaces (S,H) of genus g such that

$$\Lambda^r_{g,d}=\begin{array}{c|cc} & H & L \\ \hline H & 2g-2 & d \\ L & d & 2r-2\end{array}$$

admits a primitive embedding in $\operatorname{Pic}(S)$ preserving H.

Proposition: Let $(S,H)\in\mathcal{K}^r_{g,d}$ and let $C\in|H|$ be a smooth irreducible curve. If L and $H-L$ are basepoint free, $r\geq 2$, and $0<d\leq g-1$, then $L\otimes\mathcal{O}_C$ is a g^r_d.

Distinguishing Brill–Noether Loci and Lifting g^r_ds

Our strategy is to show that a curve on a very general polarized K3 surface in $\mathcal{K}^r_{g,d}$ admits a g^r_d, but no other expected maximal $g^{r'}_{d'}$. We do this by studying the lifting of line bundles on polarized K3 surfaces.

Donagi–Morrison Conjecture [1, 3]: Let (S,H) be a polarized K3 surface and $C\in|H|$ be a smooth irreducible curve of genus ≥ 2. Suppose A is a complete basepoint free g^r_d on C such that $d\leq g-1$ and $\rho(g,r,d)<0$. Then there exists a line bundle $M\in\operatorname{Pic}(S)$ adapted to $|H|$ such that

- $|A|$ is contained in the restriction of $|M|$ to C, and
- $\gamma(M\otimes\mathcal{O}_C)\leq\gamma(A)$.

Donagi and Morrison verified the conjecture for $r=1$, and Lelli-Chiesa verified it for $r=2$ [1, 2], she also verified it under a technical hypothesis that the pair (C,A) do not have any *unexpected secant varieties up to deformation*[3].

By considering curves $C\in\mathcal{M}^r_{g,d}$ that lie on a K3 surface S with $\operatorname{Pic}(S)=\Lambda^r_{g,d}$, and showing that $\langle H,M\rangle\not\subseteq\operatorname{Pic}(S)$, the Donagi–Morrison conjecture, and codimension considerations implies the maximal Brill–Noether loci conjecture up to genus 88. After that, there are exceptions, though we expect them to be sparse.

Genus 14

The expected maximal BN loci are $\mathcal{M}^1_{14,8}$, $\mathcal{M}^2_{14,11}$, and $\mathcal{M}^3_{14,13}$. Work of Lelli-Chiesa shows that $\mathcal{M}^3_{14,13}\not\subseteq\mathcal{M}^2_{14,11}$. Recent work on Brill–Noether theory for curves of fixed gonality shows that $\mathcal{M}^1_{14,8}$ is maximal. Moreover, using Lelli-Chiesa's lifting results, it can be shown that $\mathcal{M}^2_{14,11},\mathcal{M}^3_{14,13}\not\subseteq\mathcal{M}^1_{14,8}$. It remains to find a curve with a g^2_{11} that does not admit a g^3_{13}.

Lifting g^3_ds

Theorem: Let (S,H) be a polarized K3 surface of genus $g\neq 2,3,4,8$, and $C\in|H|$ a smooth irreducible curve of Clifford index $\gamma(C)$. Suppose that S has no elliptic curves and $d<\frac{5}{4}\gamma(C)+6$, then the Donagi–Morrison conjecture holds for any g^3_d on C.

We prove a slightly more refined version, replacing the hypothesis on non-existence of elliptic curves with an explicit dependence on the Picard lattice of S.

Proof Idea

Let A be a line bundle of type g^3_d on $C\in|H|$. If $\rho(g,r,d)<0$, then $E_{C,A}$ is not stable. To obtain a Donagi–Morrison lift of A, we want to show that $E_{C,A}$ has a maximal destabilizing subline bundle. To do this, we find lower bounds on d whenever $E_{C,A}$ has a different destabilizing subsheaf by analyzing the Harder–Narasimhan and Jordan–Hölder filtrations.

Lazarsfeld–Mukai Bundles

We define a bundle $F_{C,A}$ on S via the short exact sequence

$$0\longrightarrow F_{C,A}\longrightarrow H^0(C,A)\otimes\mathcal{O}_S\xrightarrow{ev}\iota_*(A)\longrightarrow 0.$$

Dualizing gives $E_{C,A}=F^\vee_{C,A}$ (the LM bundle associated to A on C) sitting in the short exact sequence

$$0\longrightarrow H^0(C,A)^\vee\otimes\mathcal{O}_S\longrightarrow E_{C,A}\longrightarrow\iota_*(\omega_C\otimes A^\vee)\longrightarrow 0;$$

The LM bundle $E_{C,A}$ is like a lift of A to a vector bundle on S.

Let $E_{C,A}$ be a LM bundle associated to a basepoint free line bundle A of type g^r_d on $C\subset S$, then:

- $c_1(E_{C,A})=[C]$ and $c_2(E_{C,A})=\deg(A)$;
- $\operatorname{rk}(E_{C,A})=r+1$ and $E_{C,A}$ is globally generated off the base locus of $\iota_*(\omega_C\otimes A^\vee)$;
- $h^0(S,E_{C,A})=h^0(C,A)+h^0(C,\omega_C\otimes A^\vee)=2r+1+g-d=g-(d-2r)+1$;
- $h^1(S,E_{C,A})=h^2(S,E_{C,A})=0$, $h^0(S,E^\vee_{C,A})=h^1(S,E^\vee_{C,A})=0$;
- $\chi(F_{C,A}\otimes E_{C,A})=2(1-\rho(g,r,d))$.

LM bundles are useful for lifting g^r_ds. In fact, if there is a nontrivial $N\in\operatorname{Pic}(S)$ with $h^0(S,N)\neq 0$, $h^1(S,N)=0$, and an injection $N\hookrightarrow E_{C,A}$, then the Donagi Morrison conjecture holds with $L=\mathcal{O}_S(C)\otimes N^\vee$.[3]

Stability of Sheaves on K3 Surfaces

The *slope* of E is $\mu(E)=\frac{c_1(E).H}{\operatorname{rk}(E)}$. A torsion-free coherent sheaf is called slope stable or μ-stable (μ-semistable) if $\mu(F)<\mu(E)$ (respectively, $\mu(F)\leq\mu(E)$) for all coherent sheaves $F\subsetneq E$ with $0<\operatorname{rk}(F)<\operatorname{rk}(E)$.

Every torsion-free coherent sheaf E has a unique Harder–Narasimhan filtration, which is an increasing filtration

$$0=HN_0(E)\subset HN_1(E)\subset\cdots\subset HN_\ell(E)=E,$$

such that the factors $gr^{HN}_i(E)=HN_i(E)/HN_{i-1}(E)$ for $1\leq i\leq\ell$ are torsion free semistable sheaves with $\mu(gr^{HN}_1(E))>\mu(gr^{HN}_2(E))>\cdots>\mu(gr^{HN}_\ell(E))$.

Likewise, every semistable sheaf E has a Jordan–Hölder filtration, which is an increasing filtration with stable factors all of slope $\mu(E)$.

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References

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