On the algebraic decomposition of a centralizer algebra of the hyperoctahedral group

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ABSTRACT. In this paper we give a combinatorial rule for the decomposition of tensor powers of the signed permutation representation of the hyperoctahedral group. We then use this rule to describe the Bratteli diagram of a centralizer algebra of this group over k-th tensor space. We show that a basis for this algebra can be described completely in terms of set partitions and we give a set of generators and relations.

INTRODUCTION

In [**J**] Jones has given a description of the centralizer algebra $\mathrm{End}_{S_n}(V^{\otimes k})$ where $V = \mathbb{C}^n$ and S_n (symmetric group) acts by permutations on V and it acts diagonally on $V^{\otimes k}$. This algebra was independently introduced by Martin [**Ma**] and named the Partition algebra. The main motivation for studying the partition algebra is in generalizing the Temperley-Lieb algebras and the Potts model in statistical mechanics. A survey on the Partition algebra has been written by Halverson and Ram [**HR**].

The main objective of this paper is to study a family of subalgebras of the Partition algebra. In this paper we look at the corresponding centralizer algebras of the hyperoctahedral group. This group is the wreath product of the cyclic group of order 2 and the symmetric group, which we will denote by $G_n := \mathbb{Z}/2\mathbb{Z} \,\mathrm{wr}\, S_n$.

Let $V = \mathbb{C}^n$, then G_n acts on this vector space via signed permutations. The semisimple decomposition of the algebra $\mathrm{End}_{G_n}(V^{\otimes k})$ can be obtained by decomposing the tensor product $V^{\otimes k}$ in terms of simple G_n-modules by the double centralizer theory. In this paper we prove the decomposition of the G_n-module $V^{\otimes k}$ in terms of irreducibles. Since the representations of G_n are indexed by ordered pairs of Young diagrams, we obtained an indexing set for the irreducible representations of $\mathrm{End}_{G_n}(V^{\otimes k})$ in terms of ordered pairs of Young diagrams. This decomposition rule also yields a purely combinatorial rule in terms of pairs of Young diagrams for the restriction and induction of simple modules of this centralizer algebras. This branching rule allows us to write the Bratteli diagram of these centralizer algebras.

1991 *Mathematics Subject Classification.* Primary 16G99, 05E05; Secondary 20F55, 16S99.

Key words and phrases. Centralizer algebra; Partition algebra; Hyperoctahedral group; Coxeter group; Bratteli diagram.

Partially supported by the Wilson Foundation.

We also give a linear basis of $\mathrm{End}_{G_n}(V^{\otimes k})$ and a description in terms of set partition diagrams. This basis has already appeared in [**T**], we include a proof just for completeness. The generators of this algebra are the Temperley-Lieb generators e_i, the simple transpositions s_i of S_n and some of the generators of the Partition algebra b_i and the identity. These generators have been obtained in [**T**] the set of relations satisfied by these generators are not stated anywhere as far as the author knows; however they can be easily deduced from [**BJ, HR, K, We**].

Kosuda [**K**] has studied the subalgebra of the partition algebra generated by the s_i's and the b_i's only. He has named this algebra the Party algebra. In [**K**] the Bratteli diagram of the sequence of Party algebras has been defined by example. From this example one can deduce the decomposition rules for the Party algebra. For the generic Party algebra the irreducible representations are indexed by sequences of partitions. The rule we have obtained in this paper is similar to the rule obtained in [**K**] since the Party algebra is a subalgebra of $\mathrm{End}_{G_n}(V^{\otimes k})$.

The generators b_i and e_i generate a subalgebra isomorphic to an algebra studied by Bisch and Jones [**BJ**] with relation to intermediate subfactors, the subalgebra is called the bicolored Fuss-Catalan algebra, so in some sense the centralizer algebra $\mathrm{End}_{G_n}(V^{\otimes k})$ is a bicolored analogue of the Brauer algebra where the bicolored Fuss-Catalan algebra replaces the Temperley-Lieb algebra.

This paper is organized as follows: In Section 1 we give definitions and notation that will be useful in the rest of the paper. In Section 2 we prove the branching rule in terms of Schur functions. In Section 3 we define the algebra $\mathrm{End}_{G_n}(V^{\otimes k})$ and describe a linear basis as well as generators and relations. In Section 4 we use the main theorem in Section 2 to construct the Bratteli diagram of $\mathrm{End}_{G_n}(V^{\otimes k})$.

Acknowledgement. I would like to thank T. Halverson for telling me that the basis in Sec. 3.1 has already appeared in [**T**] and also for bringing to my attention the work of M. Kosuda [**K**].

1. Preliminaries

1.1. Integer partitions and symmetric functions. A *partition* of a nonnegative integer n is a sequence of nonnegative integers $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ such that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_\ell \geq 0$ and $|\lambda| = \alpha_1 + \alpha_2 + \cdots + \alpha_\ell = n$. The nonzero α_i's are called the *parts* of α and the number of nonzero parts is called the *length* of α denoted by $\ell(\alpha)$. The notation $\alpha \vdash n$ means that α is a partition of n.

A *Young diagram* is a pictorial representation of a partition α as an array of n boxes with α_1 boxes in the first row, α_2 boxes in the second row, and so on. We count the rows from top to bottom. We shall denote the Young diagram and the partition by the same symbol α. We use the Young diagram interchangeably with the word partition. We denote by $\emptyset$ the empty partition and $|\emptyset| = 0$.

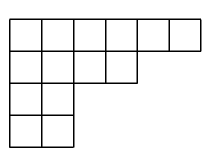


Figure 1. $\lambda = (6, 4, 2, 2), \quad \ell(\lambda) = 4, \quad |\lambda| = 14$

A *double partition* of size n, (α, β) is an ordered pair of partitions α and β such that $|(\alpha, \beta)| = |\alpha| + |\beta| = n$. A double partition corresponds to a double pair of Young diagrams in the obvious way.

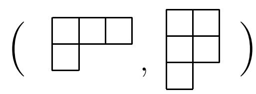


Figure 2. $(\alpha, \beta) = ([3, 1], [2, 2, 1])$ and $|(\alpha, \beta)| = 9$.

If λ and μ are two partitions with $|\mu| \leq |\lambda|$, then we write $\lambda \subset \mu$ if $\mu_i \leq \lambda_i$ for all i. In this case we say that μ is contained in λ. If $\mu \subset \lambda$ then the set-theoretic difference $\mu - \lambda$ is called a *skew diagram*, λ/μ.

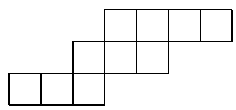


Figure 3. λ/μ for $\lambda = (7, 5, 3)$ and $\mu = (3, 2)$.

For any positive integer N, let $x = (x_1, x_2, \ldots, x_N)$ be an independent set of commuting variables. The symmetric group S_N acts on the ring of polynomials $\mathbb{C}[x_1, \ldots, x_N]$ by permuting the variables. A polynomial is called *symmetric* if it is invariant under this action. The ring of symmetric polynomials is denoted by $\Lambda_N := \mathbb{C}[x_1, \ldots, x_N]^{S_N}$. Λ_N is a graded ring: we have $\Lambda_N = \bigoplus_{k \geq 0} \Lambda_N^k$, where Λ_N^k contains the homogeneous symmetric polynomials of degree k and the zero polynomial.

Let $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_N^{\lambda_N}$, assume $N \geq |\lambda|$ and $\lambda_i = 0$ if the number of parts in λ is less than N. Let $\delta = (N-1, N-2, \ldots, 1, 0)$. We define the following determinant:

$$a_{\lambda+\delta} = \det(x_i^{\lambda_j + N - j}).$$

This determinant is divisible by the Vandermonde determinant $a_\delta = \det(x_i^{N-j})$. In this setting the symmetric *Schur function* is defined by

$$s_\lambda(x) = s_\lambda(x_1, x_2, \ldots, x_N) := \frac{a_{\lambda+\delta}}{a_\delta}.$$

It is well-known that $\{s_\lambda \,|\, |\lambda| = k\}$ is a basis of Λ_N^k. In this paper we will also use another basis of Λ_N^k known as the *power symmetric functions* defined as follows: for any nonnegative integer r, let $p_r := x_1^r + x_2^r + \cdots + x_N^r$. Then for any partition λ, we have

$$p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_{\ell(\lambda)}}.$$

1.2. Hyperoctahedral group. In this section we give a brief summary on results about the hyperoctahedral group G_n that will be useful in the remainder of this paper. For proofs and details see [**JK**]. The hyperoctahedral group is the wreath product of the cyclic group of order 2 with S_n (symmetric group): $G_n := (\mathbb{Z}/2\mathbb{Z}) \operatorname{wr} S_n$.

G_n is generated by $t, s_1, \ldots, s_{n-1}$ satisfying the following relations:

(1) $t^2 = 1$;

(2) $s_i^2 = 1$, $1 \leq i \leq n-1$;

(3) $ts_1ts_1 = s_1ts_1t$;

(4) $s_is_j = s_js_i$, $|i-j| > 1$;

(5) $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$.

The irreducible representations of G_n are indexed by double partitions or ordered pairs of Young diagrams (α, β) of total size n, i.e. $|\alpha| + |\beta| = n$. It is also known that the dimension of the irreducible representation indexed by (α, β) is given by

$$f^{(\alpha,\beta)} = \binom{n}{|\alpha|} f^\alpha f^\beta$$

where $f^\gamma = \frac{|\gamma|!}{\prod\limits_{(i,j)\in\gamma} h(i,j)}$ and $h(i,j) = \gamma_i + {\gamma_j}' - i - j + 1$, where the prime indicates the length of the j'th column of the Young diagram γ.

1.3. The Characteristic Map. In this section we relate the characters of G_n with symmetric functions. We will use this relationship to prove the tensor product decomposition of the natural representation in terms of irreducibles. The details and proofs of the results quoted in this section can be found in [**M**], [**S**], and [**B**].

Let $\mathcal{C}(G_n)$ be the algebra of class functions over the complex numbers and let

$$\Lambda(G_n) := \bigoplus_{m_1+m_2=n} \Lambda^{m_1}(x) \otimes \Lambda^{m_2}(y)$$

where $\Lambda^m(x)$ is the space of homogeneous symmetric functions of degree m. Let $C_{(\alpha,\beta)}$ denote the conjugacy class of G_n indexed by (α, β). Define the class functions $1_{(\alpha,\beta)}(\sigma) = \chi(\sigma \in C_{(\alpha,\beta)})$, that is

$$1_{(\alpha,\beta)}(\sigma) = \begin{cases} 1 & \text{if } \sigma \in C_{(\alpha,\beta)} \\ 0 & \text{otherwise.} \end{cases}$$

It is well-known that the set $\{1_{(\alpha,\beta)} \,|\, |(\alpha,\beta)| = n\}$ is a basis of $\mathcal{C}(G_n)$. Thus, one can define the characteristic map on this basis as follows

$$\begin{array}{rcl} ch : \mathcal{C}(G_n) & \longrightarrow & \Lambda(G_n) \\ 1_{(\alpha,\beta)} & \longrightarrow & \dfrac{1}{Z_{(\alpha,\beta)}} p_\alpha[x]p_\beta[y] \end{array}$$

where $Z_{(\alpha,\beta)} = z_\alpha z_\beta$ and $z_\lambda = \prod_{i\geq 1} i^{n_i} n_i!$ with n_i being the number of parts of λ equal to i. Using this map one can write the characteristic for the irreducible characters of G_n in terms of Schur functions. The proof of the following proposition is found in [**B**] and [**M**].

Proposition 1.1. *Let $\chi^{(\alpha,\beta)}$ denote the irreducible character of G_n indexed by (α, β). Then its characteristic is given as follows:*

$$ch(\chi^{(\alpha,\beta)}) := S_{(\alpha,\beta)}(x,y) = s_\alpha[x+y]s_\beta[x-y]$$

where $s_\alpha[x+y] = \sum\limits_{\gamma\subset\alpha} s_\gamma(x)s_{\alpha/\gamma}(y)$ and $s_\beta[x-y] = \sum\limits_{\delta\subset\beta} s_\delta(x)(-1)^{|\beta/\delta|}s_{(\beta/\delta)'}(y)$, and the prime denotes the conjugate partition.

2. Branching Rule

Let V be the signed permutation representation of the hyperoctahedral group. It is known that this representation is irreducible and that it is indexed by $([n-1],[1])$. In this section we prove via elementary methods the decomposition of the G_n-module $V^{\otimes k}$, for any positive integer k, in terms of irreducible representations of G_n. The methods employed are basic properties of symmetric functions found in [**M**] and known results about the inner product of symmetric functions.

Let $V_{(\alpha,\beta)}$ and $V_{(\lambda,\mu)}$ be two irreducible representations of G_n with characters $\chi^{(\alpha,\beta)}$ and $\chi^{(\lambda,\mu)}$ respectively, then let G_n acts diagonally on $V_{(\alpha,\beta)} \otimes V_{(\lambda,\mu)}$. The character of this representation is $\chi^{(\alpha,\beta)}\chi^{(\lambda,\mu)}$, the point-wise product defined by $\chi^{(\alpha,\beta)}\chi^{(\lambda,\mu)}(w) = \chi^{(\alpha,\beta)}(w)\chi^{(\lambda,\mu)}(w)$ for all $w \in G_n$. Computing the decomposition of the G_n-module $V_{(\alpha,\beta)} \otimes V_{(\lambda,\mu)}$ is equivalent to decomposing $\chi^{(\alpha,\beta)}\chi^{(\lambda,\mu)}$. It is known that

$$ch(\chi^{(\alpha,\beta)}\chi^{(\lambda,\mu)}) = S_{(\alpha,\beta)}(x,y) * S_{(\lambda,\mu)}(x,y),$$

where the $*$ denotes the inner product (or Kronecker product) of symmetric functions. Therefore, computing the inner product of the corresponding characteristics yields the decomposition of the tensor product. We will use this method to prove the decomposition of $V^{\otimes k}$.

In the proof of the following lemma and theorem we will need the following results on symmetric functions. The proofs of these facts can be found in [**M**]. In the following f, g, and h denote symmetric functions, $\perp$ denotes the skew operator, i.e. $s_\lambda^\perp(s_\mu) = s_{\mu/\lambda}$, and the $*$ denotes the inner or Kronecker product of symmetric functions.

(1) $(fg) * s_\lambda = \sum_{\mu\subset\lambda}(f * s_{\lambda/\mu})(g * s_\mu)$.

(2) $s_{n-1,1} * s_\lambda = s_\square s_{\lambda/\square} - s_\lambda$, where $|\lambda| = n$.

(3) $f^\perp(gh) = \sum_i a_i^\perp(g)b_i^\perp(h)$ where $\Delta(f) = \sum a_i \otimes b_i$.

(4) $s_\lambda^\perp s_\mu^\perp = s_\mu^\perp s_\lambda^\perp$.

(5) $(s_\lambda(x)s_\mu(y)) * (s_\nu(x)s_\eta(y)) = (s_\lambda(x) * s_\nu(x))(s_\mu(y) * s_\eta(y))$, if $|\lambda| = |\nu|$ and $|\mu| = |\eta|$ or zero otherwise.

(6) $f * (g+h) = f * g + f * h$.

Throughout this paper λ^+ denotes a Young diagram obtained from λ by adding one box. And λ^- will denote a Young diagram obtained from λ by removing a box. And a sum over λ^+ or λ^- means the sum over all possible Young diagrams of this kind. By Proposition 1.1 we have that

$$S_{([n-1],[1])}(x,y) = \\ = \sum_{m=0}^{n-1}[s_{m+1}(x) + s_{m,1}(x)]s_{n-m-1}(y) - s_m(x)[s_{n-m}(y) + s_{n-m-1,1}(y)].$$

We now prove some identities that we will use in the proof of the next theorem.

Lemma 2.1. (a) $\sum_{\gamma\subset\alpha} s_{\gamma/\square}(x)s_{\alpha/\gamma}(y) = \sum_{\alpha^-}\sum_{\gamma\subset\alpha^-} s_\gamma(x)s_{\alpha^-/\gamma}(y)$

(b) $\sum_{\gamma\subset\alpha} s_\gamma(x)s_\square(x)s_{\alpha/\gamma}(y) = \sum_{\alpha^+}\sum_{\square\subset\gamma\subset\alpha^+} s_\gamma(x)s_{\alpha^+/\gamma}(y) - \sum_{\square\subset\gamma\subset\alpha} s_\gamma(x)s_\square(y)s_{\alpha/\gamma}(y)$

(c) $\sum_{\delta\subset\beta}(-1)^{|\beta/\delta|}s_{\gamma/\square}(x)s_{(\beta/\delta)'}(y) = \sum_{\beta^-}\sum_{\delta\subset\beta^-} s_\delta(x)(-1)^{\beta^-/\delta}s_{(\beta^-/\delta)'}(y)$

(d) $\sum_{\delta\subset\beta} s_\square(x)s_\delta(x)(-1)^{|\beta/\delta|}s_{(\beta/\delta)'}(y) = \sum_{\beta^+}\sum_{\delta\subset\beta^+} s_\delta(x)(-1)^{|\beta^+/\delta|}s_{(\beta^+/\delta)'}(y)$
$-\sum_{\delta\subset\beta} s_\delta(x)(-1)^{|\beta/\delta|-1}s_\square(y)s_{(\beta/\delta)'}(y)$

Proof. (a) The left-handside of (a) can be written as follows:

$$\begin{aligned}
\text{LHS} &= \sum_{\gamma\subset\alpha}\sum_{\gamma^-} s_{\gamma^-}s_{\alpha/\gamma}\\
&= \sum_{\substack{\gamma\subset\alpha\\ \gamma\neq\alpha}} s_\gamma(x)\sum_{\gamma^+} s_{\alpha/\gamma^+}(y)\\
&= \sum_{\substack{\gamma\subset\alpha\\ \gamma\neq\alpha}} s_\gamma s_{(\alpha/\gamma)/\square}(y)\\
&= \sum_{\alpha^-}\sum_{\gamma\subset\alpha^-} s_\gamma(x)s_{\alpha^-/\gamma}(y)
\end{aligned}$$

The first equality follows by expanding $s_{\gamma/\square}$, the second by rearranging the order of summation, and the last equation follows from the commutativity of the skew operation.
(b)

$$\begin{aligned}
\text{LHS} &= \sum_{\gamma\subset\alpha}\sum_{\gamma^+} s_{\gamma^+}(x)s_{\alpha/\gamma}(y)\\
&= \sum_{\square\subset\gamma\subset\alpha^+} s_\gamma(x)\sum_{\gamma^-\subset\alpha} s_{\alpha/\gamma^-}(y)\\
&= \sum_{\square\subset\gamma\subset\alpha^+} s_\gamma(x)[(\sum_{\alpha^+} s_{\alpha^+/\gamma}(y)) - s_\square(y)s_{\alpha/\gamma}(y)]\\
&= \sum_{\alpha^+}\sum_{\square\subset\gamma\subset\alpha^+} s_\gamma s^{\alpha^+/\gamma}(y) - \sum_{\square\subset\gamma\subset\alpha} s_\gamma(x)s_\square(y)s_{\alpha/\gamma}(y)
\end{aligned}$$

The first equality is just a change in the order of summation. The second equality is a special case of equation (3) above, where $f = s_\gamma(y)$, $g = s_\square(y)$ and $h = s_\alpha(y)$.

The proof of (c) and (d) are done in a similar way as the proofs of (a) and (b) respectively. $\square$

We now state the main theorem of this section which will allow us to decompose the tensor product of the sign permutation module with any other G_n-module.

Theorem 2.1.

$$S_{([n-1],[1])}(x,y) * S_{(\alpha,\beta)}(x,y) = \sum_{\alpha^+,\beta^-} S_{(\alpha^+,\beta^-)}(x,y) + \sum_{\alpha^-,\beta^+} S_{(\alpha^-,\beta^+)}(x,y)$$

where α^+ is a diagram obtained by adding one box to α, and α^- is obtained by removing one box from α, and similarly for β.

Proof. After substitution and distributing by relations (1), (5) and (6) above we have that the lefthand side of the equation of the statement of the theorem can be simplified to:

$$\sum_{m=1}^{n}\sum_{\substack{\gamma\subset\alpha,\delta\subset\beta\\|\alpha|+|\beta|=m}}[(s_m(x)+s_{m-1,1}(x))*(s_\gamma(x)s_\delta(x))][s_{n-m}(y)*((-1)^{|\beta/\delta|}s_{\alpha/\gamma}(y)s_{(\beta/\delta)'}(y))]$$

$$-$$

$$\sum_{m=1}^{n}\sum_{\substack{\gamma\subset\alpha,\delta\subset\beta\\|\alpha|+|\beta|=m}}s_m(x)*(s_\gamma(x)s_\delta(x))[(s_{n-m}(y)+s_{n-m-1,1}(y))*((-1)^{|\beta/\delta|}s_{\alpha/\gamma}(y)s_{(\beta/\delta)'}(y))$$

$$=\sum_{m=1}^{n}\sum_{\substack{\gamma\subset\alpha,\delta\subset\beta\\|\alpha|+|\beta|=m}}[s_{\gamma/1}(x)s_1(x)s_\delta(x)+s_\gamma(x)s_{\delta/1}(x)s_1(x)][s_{\alpha/\gamma}(y)(-1)^{|\beta/\delta|}s_{(\beta/\delta)'}(y)]-$$

$$\sum_{m=0}^{n-1}\sum_{\substack{\gamma\subset\alpha,\delta\subset\beta\\|\gamma|+|\beta|=m}}[s_\gamma(x)s_\delta(x)](-1)^{|\beta/\delta|}[s_{(\alpha/\gamma)/1}(y)s_1(y)s_{\beta/\delta}(y)+s_{\alpha/\gamma}(y)s_{(\beta/\delta)'/1}(y)s_1(y)]$$

The equality follows from equations (2) and (5). Now if we replace the equations in Lemma 2.1, we will get the right-hand side of the statement of the theorem. □

The main application of Theorem 2.1 is that it allows us to decompose the G_n-module $V^{\otimes k}$ for any k, where V is the irreducible G_n-module indexed by $([n-1],[1])$. For example,

$$V\otimes V\cong V_{([n-2],[2])}\oplus V_{([n-2],[1,1])}\oplus V_{([n],\emptyset)}\oplus V_{([n-1,1],\emptyset)}.$$

We will use this rule to recursively construct the Bratteli diagram for the centralizer algebras $\mathrm{End}_{G_n}(V^{\otimes k})$.

3. Centralizer Algebra of G_n

Let $V=\mathbb{C}^n$,and let $\{v_i\,|\,1\le i\le n\}$ denote its standard basis, i.e. v_i is the vector whose i^{th} entry is 1 and the rest are zeros. G_n acts naturally on V as follows:

$$t.v_i=\begin{cases}-v_1 & \text{if } i=1,\\ v_i & \text{otherwise,}\end{cases}$$

and $\sigma.(v_i)=v_{\sigma(i)}$, for any permutation σ, since G_n is generated by t and the group of permutations.

G_n also acts diagonally on $V^{\otimes k}$ for any positive integer k,

$$g.(v_{i_1}\otimes\cdots\otimes v_{i_k})=g.v_{i_1}\otimes\cdots\otimes g.v_{i_k}$$

for any $g\in G_n$. Thus, $V^{\otimes k}$ is a G_n-module and therefore we have the fact that the endomorphism algebra $\mathrm{End}_{G_n}(V^{\otimes k})$ is the commutant algebra, that is

$$\mathrm{End}_{G_n}(V^{\otimes k})=\{X\,|\,Xg=gX \text{ for all } g\in G_n\}$$

Remark: Notice that since S_n is a subgroup of G_n, it follows that the centralizer algebra $\mathrm{End}_{G_n}(V^{\otimes k})$ is a subalgebra of $\mathrm{End}_{S_n}(V^{\otimes k})$.

3.1. A basis for $A_k(n) := \mathbf{End}_{G_n}(V^{\otimes k})$. In this subsection we describe a basis for $A_k(n) := \mathrm{End}_{G_n}(V^{\otimes k})$. This basis is the same as in [**T**]. For any element $X \in A_k(n)$ we denote by $(X_{a_1,\cdots a_k}^{b_1,\cdots b_k})$, $1 \le a_i, b_i \le n$, its matrix of coefficients with respect to the basis $\{v_{i_1} \otimes \cdots \otimes v_{i_k} \,|\, 1 \le i_j \le n\}$ of $V^{\otimes k}$. Set $[n] := \{1, 2, \ldots, n\}$.

Lemma 3.1. *$X \in A_k(n)$ if and only if $X_{a_1,\cdots a_k}^{b_1,\cdots b_k} = X_{\sigma(a_1),\cdots,\sigma(a_k)}^{\sigma(b_1),\cdots,\sigma(b_k)}$ for any permutation σ and $|\{i \,|\, a_i = j\}| \equiv |\{i \,|\, b_i = j\}|$ mod 2 for all $j = 1, \ldots, n$ and $i = 1, \ldots, k$.*

Proof. For $X \in A_k(n)$ and $\sigma \in S_n$ we have

$$\sigma^{-1} X \sigma(v_{a_1} \otimes \cdots \otimes v_{a_k}) = \sum_{b_1,\ldots b_k \subseteq [n]} X_{\sigma(a_1),\ldots\sigma(a_k)}^{\sigma(b_1)\ldots\sigma(b_k)} v_{b_1} \otimes \cdots \otimes v_{b_k}.$$

Now for the generator t we have

$$t^{-1} X t(v_{a_1} \otimes \cdots v_{a_k}) = \sum_{b_1,\ldots,b_k \subseteq [n]} (-1)^{|\{i \,|\, a_i = 1\} - |\{i \,|\, b_i = 1\}|} X_{a_1\ldots a_k}^{b_1,\ldots,b_k} v_{b_1} \otimes \cdots \otimes v_{b_k},$$

since $(-1)^{|\{i \,|\, a_i=1\}| - |\{i \,|\, b_i=1\}|} = 1$ if and only if $|\{i \,|\, a_i = 1\} - |\{i \,|\, b_i = 1\}| \equiv 0$ mod 2 we have the second condition of the statement of the lemma. The result follows in general by linear independence and since G_n is generated by t and the permutation group. $\square$

This lemma says that X commutes with the action of G_n on $V^{\otimes k}$ if and only if the matrix entries are equal on G_n-orbits on the $2k$ Cartesian product, $\{1, 2, \ldots, n\}^{\times 2k}$. These orbits are in $1 - 1$ correspondence with set partitions, $B = \{B_1, \ldots B_s\}$ of $\{1, 2, \ldots, 2k\}$, such that all blocks have even cardinality.

Every set partition $B = \{B_1, B_2, \ldots, B_s\}$ has at most n blocks. Notice that each set partition will give rise to an equivalence relation, i.e. two elements $i, j \in \{1, 2, \ldots, 2k\}$ are equivalent if and only if they belong to the same block if and only if $a_i = a_j$, where we have relabeled $b_l = a_{l+k}$ for convenience.

For each set partition B of $[2k]$ we define an element $T_B \in A_k(n)$ by the matrix

$$(T_B)_{a_1\ldots a_k}^{a_{k+1}\ldots a_{2k}} = \begin{cases} 1 & \text{if } a_i = a_j \text{ if and only if } i \text{ and } j \text{ are in the same block.} \\ 0 & \text{otherwise.} \end{cases}$$

We have the following proposition.

Proposition 3.1. *Let $\mathcal{B}$ be the set of all set partitions of $[2k]$ such that for any set partition $B = \{B_1, \ldots, B_s\}$ in $\mathcal{B}$ all blocks B_i have even cardinality. Then the set $\{T_B \,|\, B \in \mathcal{B}\}$ form a basis for $A_k(n)$ if we exclude any zero T_B.*

Proof. Notice that T_B is zero if the number of blocks is greater than n, thus as long as $2k < n$ the T_B will be nonzero. By definition, the set $\{T_B \,|\, B \in \mathcal{B}\}$ spans $A_k(n)$. Notice that another way to write T_B is in terms of the matrices $E_{a_1\ldots a_k}^{a_{k+1}\ldots a_{2k}}$ which are $n^k \times n^k$ matrices with a 1 in the $(a_1 \ldots a_k), (a_{k+1} \ldots a_{2k})$ position and zero everywhere else. Then we have

$$T_B = \sum E_{a_1\ldots a_k}^{a_{k+1}\ldots a_{2k}}$$

where the sum is over all $1 \le a_1, \ldots, a_{2k} \le n$ with the condition that $a_i = a_j$ if and only if i and j are in the same block in B.

To show linear independence, assume that $\sum_B c_B T_B = 0$ (the zero matrix). Consider the $(i_1, i_2, \ldots, i_k), (i_{k+1}, \ldots, i_{2k})$ entry in the left-hand side of this equation. This entry is equal to $\sum_D c_D$, where D is a set partition finer than or equal the set partition, B, (i.e. $D_i \subset B_j$ for some j) defined by the $2k$-tuple $(i_1, \ldots, i_k), (i_{k+1}, \ldots, i_{2k})$. This implies that $c_{\{j_1,j_2\}\cdots\{j_k,j_{2k}\}} = 0$, since a set partition into blocks of size 2 is the smallest set partition in $\mathcal{B}$. Notice that by induction for any arbitrary set partition $B \in \mathcal{B}$ we can write c_B as the sum of coefficients c_D where the D's are set partitions with only blocks of size 2. Thus, $c_B = 0$ for all $B \in \mathcal{B}$. Thus $\{T_B \mid B \in \mathcal{B}\}$ forms a basis. □

Corollary 3.1. *The dimension of End($V^{\otimes k}$) is the number d_k of set partitions of a $2k$-element set into even blocks. This dimension satisfies the recurrence $d_k = \sum_{j=1}^{k} \binom{2k-1}{2j-1} d_{k-j}$ and has exponential generating function exp(cosh(x) $-$ 1).*

For a proof of the second statement of the corollary see [**C**]. The first five dimensions are 1, 4, 31, 379, 6556.

3.2. Partition diagrams. A set partition $B = \{B_1, B_2, \ldots B_s\}$ of $[2k]$ can be represented by a class of simple graphs with vertex set $\{1, 2, \ldots, 2k\}$ arranged in two rows and such that two vertices x and y are in the same block in B if and only if they are connected by a path in the graph. Thus, every block defines a connected component in the graph. We think of two graphs with the same connected components as equivalent, see Figure 4 for an example of a partition diagram.

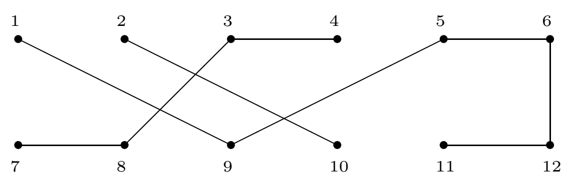


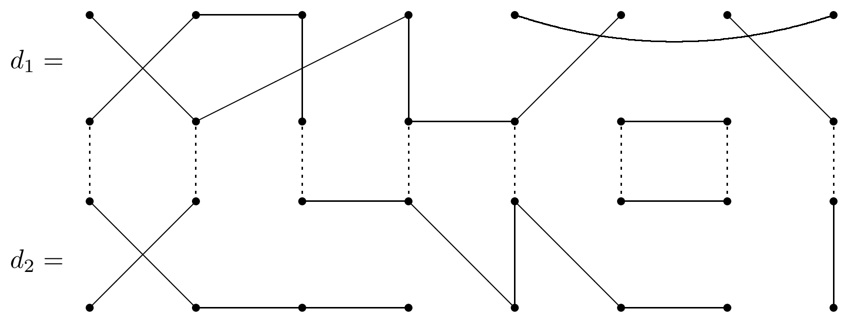
Figure 4. A partition diagram corresponding to $\{1, 5, 6, 9, 11, 12\}\{2, 10\}\{3, 4, 7, 8\}$

The equivalence classes of graphs corresponding to set partitions are called *partition diagrams* or *k-partition diagrams* if it is necessary to specify the number of vertices. One can define a multiplication of partition diagrams as follows.

Let x be any indeterminate. We multiply two k-partition diagrams d_1 and d_2 to obtain the product $d_1 d_2$ as follows:

(1) Stack d_1 on top of d_2 so that the bottom vertices of d_1 are identified with the top vertices of d_2. This results in a graph with $3k$ vertices, top, middle and bottom.
(2) Let c be the number of connected components that contain only vertices from the middle row. Remove these blocks from the product graph.
(3) Create a new partition diagram d_3 that eliminates middle vertices, but connecting any top vertex with a bottom vertex if they were connected through a middle vertex.

Then, one defines $d_1 d_2 = x^c d_3$, for an example of this product see Fig. 5.



Hence,

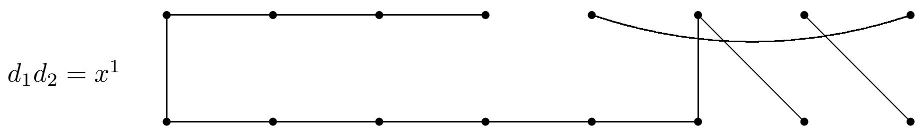


Figure 5. Product of partition diagrams.

From now on $A_k(n)$ denotes the centralizer algebra $\mathrm{End}_{G_n}(V^{\otimes k})$. We also identify the matrices T_B with their corresponding set partition diagrams.

3.3. Generators And Relations of $A_k(n)$. We introduce the following special elements of $A_k(n)$ corresponding to the following set partitions:

$$b_i = \{1, k+1\}\{2, k+2\} \dots \{i, i+1, k+i, k+i+1\} \dots \{k, 2k\};$$
$$e_i = \{1, k+1\}\{2, k+2\} \dots \{i, i+1\}\{k+i, k+i+1\} \dots \{k, 2k\};$$
$$s_i = \{1, k+1\}\{2, k+2\} \dots \{i, k+i+1\}\{i+1, k+i\} \dots \{k, 2k\},$$

where $1 \le i \le k-1$. See Fig. 6 for the corresponding partition diagrams.

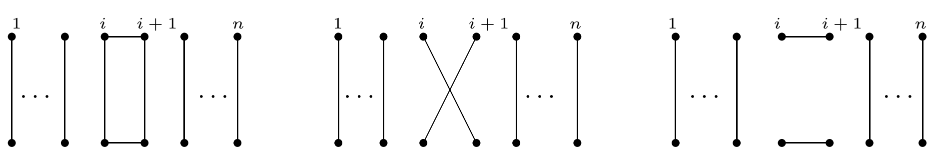


Figure 6. Generators b_i, s_i and e_i.

Proposition 3.2. *The set $\{e_i, b_i, s_i \,|\, 1 \le i \le k-1\}$ generate $A_k(n)$*

The proof of this proposition is by induction on k, and it is exactly the same as the proof of Prop. 2 in [**Ma**], therefore we omit it. This proposition has also been proved in [**T**]

Proposition 3.3. *The generators b_i, e_i and s_i, $1 \le i \le k-1$ satisfy the following relations:*

$$s_i^2 = 1, \qquad e_i^2 = ne_i, \qquad e_i s_i = s_i e_i = e_i, \qquad \textit{for } 1 \le i \le k-1;$$
$$s_i s_j = s_j s_i, \qquad s_i e_j = e_j s_i, \qquad e_i e_j = e_j e_i, \qquad \textit{for } 1 \le i \le k-1;$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad e_i e_{i+1} e_i = e_i, \quad e_{i+1} e_i e_{i+1} = e_{i+1}, \qquad \textit{for } 1 \le i \le k-2;$$

$$s_ie_{i+1}e_i = s_{i+1}e_i, \qquad e_{i+1}e_is_{i+1} = e_{i+1}s_i, \qquad \textit{for } 1 \leq i \leq k-2;$$

$$b_i^2 = b_i, \qquad b_ib_j = b_jb_i, \qquad b_ie_i = e_ib_i = e_i, \; \textit{for} \qquad 1 \leq i, j \leq k-1;$$

$$b_ie_j = e_jb_i, \qquad s_jb_i = b_is_j \qquad \textit{for } |i-j| > 1;$$

$$e_{i\pm1}b_ie_{i\pm1} = e_i, \qquad b_ie_{i\pm1}b_i = b_{i\pm1}b_i. \qquad \textit{for } 1 \leq i \leq k-1;$$

$$s_ib_i = b_is_i = b_i, \qquad s_is_{i+1}b_is_{i+1}s_i = b_{i+1}. \qquad \textit{for } 1 \leq i \leq k-1.$$

It is a straightforward exercise to check that these relations hold by using partition diagrams.

Remark: One can show that this generators and relations are a presentations for this algebra by doing word analysis. We will not carry out these computations in this paper.

4. Bratteli diagram

In this section we give an indexing set for the irreducible components of $A_k(n)$ and describe its Bratteli diagram. From this diagram one can easily read restriction and induction of the irreducibles of $A_k(n)$. We first recall the definition of a Bratteli diagram as described in [**GHJ**].

Recall that one may represent the inclusion of $A \subset B$ of multimatrix algebras (with the same unit) by a bipartite graph. The vertices in the graph are labeled by the simple summands of A and B. The number of edges joining a vertex v for A to a vertex w for B is the number of times the representation v occurs in the restriction of the representation w to A.

Given a sequence of inclusions $A_0 \subset A_1 \subset A_2 \subset \cdots$ of multimatrix algebras, one may connect the bipartite graphs describing the inclusions $A_i \subset A_{i+1}$, to obtain the *Bratteli diagram* of the sequence.

We can think of $A_{k-1}(n) \subset A_k(n)$ by identifying the elements in $A_{k-1}(n)$ with the elements of $A_k(n)$ that contain the block $\{k, 2k\}$, in other words the partition diagrams that contain a vertical line joining k and $2k$ with no other connections.

Since $A_k(n)$ is a centralizer algebra, we know by double centralizer theory that the decomposition of $V^{\otimes k}$ as a G_n-module yields the decomposition of $A_k(n)$. The vertices at the k-th level of the Bratteli diagram are indexed by the irreducible components of $V^{\otimes k}$ as a G_n-module and the number of edges joining a vertex indexed by ρ in the $(k-1)$st level to a vertex π in the kth level is given be the multiplicity of π in $V_\rho \otimes V$ as a G_n-module.

From Theorem 2.1 we have that for $n \geq 2k$ the irreducible representation of $A_k(n)$ are indexed by the following subset of ordered pairs of partitions:

$$\Gamma(k) = \{([n-j, \alpha], \beta) \,|\, \beta \vdash k-2i, \; \alpha \vdash r, \; 0 \leq r \leq i, \; 0 \leq i \leq \lfloor \frac{j}{2} \rfloor, \; j = |\alpha| + |\beta|\}$$

Observation: There is a bijection from $\Gamma(k)$ to a subset of ordered pair of Young diagrams with k or less boxes given by removing the first part of the first coordinate in the pairs in $\Gamma(k)$, i.e $([n-j, \alpha], \beta) \to (\alpha, \beta)$, where α and β satisfy the conditions in the definition of $\Gamma(k)$.

As a direct consequence of the double centralizer theory and Theorem 2.1 we have the following result.

Proposition 4.1. *The Bratteli diagram of the chain*

$$A_0(n) \subset A_1(n) \subset A_2(n) \subset A_3(n) \subset \cdots$$

is the graph where the vertices in k-th level are labeled by the elements in the set $\Gamma(k)$, $k \geq 0$, and the edges are defined as follows: a vertex (α, β) in the i-the level is joined to a vertex (λ, μ) in the $(i+1)$-st level if (λ, μ) can be obtained from the pair (α, β) by removing a box from the Young diagram in one coordinate and adding it to the Young diagram in the other coordinate.

See Fig. 7 for the first three rows of the Bratteli diagram where we have removed the first part $n - j$ from the diagram in the first coordinate as in the observation before the statement of the proposition. The 4th row of the Bratteli diagram contains 13 pairs.

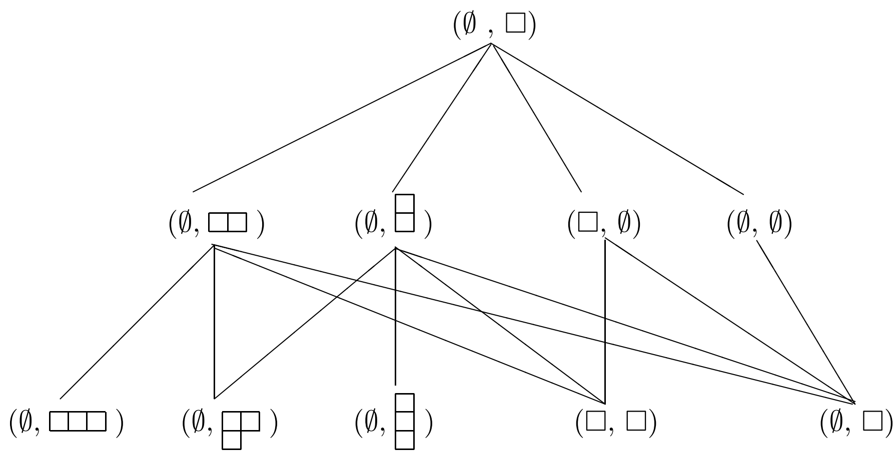


Figure 7. Bratteli diagram for the first three levels.

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