ON PARTITION ALGEBRAS FOR COMPLEX REFLECTION GROUPS

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Abstract. In this paper we study the representation theory of two partition algebras related to complex reflection groups. The colored partition algebras, $\mathcal{P}_k(n,r)$ introduced by Bloss [2] and the algebras, $\mathcal{T}_k(n,r)$ introduced by Tanabe [26]. In particular, we describe the decomposition of these algebras in terms of irreducible representations.

Introduction

The partition algebra, $\mathcal{P}_k(x)$, was introduced in the the early 1990's independently by Jones [10] and Martin [15]. The main motivation for introducing this algebra was generalizing the Temperley-Lieb algebra and the Potts model in statistical mechanics to higher dimensions. Since its introduction this algebra has been widely studied [4, 8, 9, 16, 17, 19, 18, 27] and it is for the most part well-understood. For example Martin and Woodcock [18] have extensively studied the structure of the general partition algebra $\mathcal{P}_k(\xi)$, where $\xi \in \mathbb{C}$. They have demonstrated that this algebra is semisimple whenever ξ is not an integer between 0 and $2k-1$ and they have analyzed the irreducible representations in both the semisimple and nonsemisimple cases.

Jones [10] introduced the partition algebra as a centralizer algebra of the symmetric group, S_n, over tensor space. That is, let V be the permutation representation of S_n, then the partition algebra can be defined by

$$\mathrm{End}_{S_n}(V^{\otimes k}),$$

where $V^{\otimes k}$ is an S_n-module defined via the diagonal action of S_n on $V^{\otimes k}$. In particular, he explicitly described the Schur-Weyl duality between $\mathcal{P}_k(n)$ and the symmetric group S_n. It is natural to look for a similar construction for groups other than the symmetric groups. An obvious choice are wreath products, $G \wr S_n$, where G is a finite group. In this paper we will discuss two generalizations that focus on the case that G is a cyclic group and hence $G_{n,r} := \mathbb{Z}/r\mathbb{Z} \wr S_n$. The first generalization is by Tanabe [26] and the second is by Bloss [2]. Tanabe's algebra turned out to be a subalgebra of $\mathcal{P}_k(n)$ while Bloss's algebra contains $\mathcal{P}_k(n)$ as a subalgebra. To study the representation theory of these two algebras we need information about the representation theory of the group G, in this paper we assume that G is cycic. In [11] Kosuda has studied the algebra of

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uniform block permutations, this algebra is a subalgebra of Tanabe's algebra in the case when the r in $G_{n,r}$ is chosen to be larger than n.

The objective of this paper is to study the representation theory of Tanabe's algebra, $\mathcal{T}_k(n,r)$, and the colored partition algebra of Bloss, $\mathcal{P}_k(n,r)$. The representation theory of these algebras is closely tied to the Kronecker product of characters of the group $G_{n,r}$. In the construction of both algebras we choose a $G_{n,r}$-module M and define the algebras

$$\mathrm{End}_{G_{n,r}}(M^{\otimes k}),$$

since M is a $G_{n,r}$-module, then $M^{\otimes k}$ is a $G_{n,r}$-module defined via the diagonal action of $G_{n,r}$. Using double centralizer theory one can decompose the algebra $\mathrm{End}_{G_{n,r}}(M^{\otimes k})$ in terms of irreducible representations if one knows how to decompose $M^{\otimes k}$ in terms of irreducibles.

The main result of this paper is the decomposition of the colored partition algebra, $\mathcal{P}_k(n,r)$, and the subalgebras $\mathcal{T}_k(n,r)$ in terms of irreducible representations. We describe an indexing set for the irreducible representations whenever these algebras are semisimple. We prove decomposition rules for the Kronecker product of some representations of the complex reflection groups $G_{n,r}$, and use these rules to construct the Bratteli diagram for the algebras $\mathcal{P}_k(n,r)$ and $\mathcal{T}_k(n,r)$. Our aim throughout is to give a unified approach.

This paper is organized as follows, in the first section we discuss some basic notation about partitions of numbers, and r-tuples of partitions, we also recall some of the basic facts about the group $G_{n,r}$, its irreducible representations and their relation to symmetric functions via the Frobenius characteristic map. Finally we discuss the permutation representation of $G_{n,r}$ and prove its decomposition in terms of irreducible $G_{n,r}$ representations.

In Section 2, we prove our branching rules. We use the relationship between irreducible characters and Schur functions to prove the decomposition of tensor powers of the modules used in the construction of $\mathcal{P}_k(n,r)$ and $\mathcal{T}_k(n,r)$. In Section 3 we recall some known results about the Partition algebra to motivate the results that we obtain in Section 4 on the colored partition algebra, $\mathcal{P}_k(n,r)$, and in Section 5 on the algebras $\mathcal{T}_k(n,r)$.

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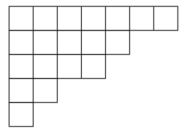
1. Preliminaries

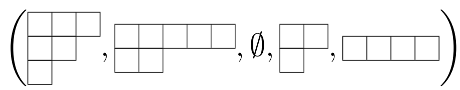
A *partition* λ of a positive integer n, denoted $\lambda \vdash n$, is a sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$. The *length* of a partition, $\ell(\lambda)$, is the number of nonzero parts of λ. Given two partitions λ and μ we say that $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i. If $\mu \subseteq \lambda$, let λ/μ denote the skew shape given by removing μ from λ. As usual we identify a partition with its corresponding

Young diagram, the left-justified array of boxes with λ_i boxes in the i-th row. For example



$$\lambda = (7, 5, 4, 2, 1),\ \ell(\lambda) = 5 \text{ and } |\lambda| = 19.$$

We will denote the empty partition or Young diagram by $\emptyset$. An r-tuple of Young diagrams $\vec{\lambda} := (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$ of total size n is an ordered sequence of length r such that each $\lambda^{(i)}$ is a Young diagram and $|\vec{\lambda}| = \sum_{i=1}^{r} |\lambda^{(i)}| = n$. For example, the following is a 5-tuple:



$$\vec{\lambda} = ((3, 2, 1), (5, 2), \emptyset, (2, 1), (4)) \text{ and } |\vec{\lambda}| = 20.$$

If $\vec{\xi} = (\xi^{(1)}, \ldots, \xi^{(r)})$ and $\vec{\nu} = (\nu^{(1)}, \ldots, \nu^{(r)})$, then we say $\vec{\xi} \subseteq \vec{\nu}$ if $\xi^{(i)} \subseteq \nu^{(i)}$ for all $i = 1, \ldots, r$. If $\vec{\xi} \subseteq \vec{\nu}$, then we let $\vec{\nu}/\vec{\xi} = (\nu^{(1)}/\xi^{(1)}, \ldots, \nu^{(r)}/\xi^{(r)})$.

For a positive integer k define the power symmetric function in the variables $x_1, x_2, \ldots$ to be

$$p_k(x_1, x_2, \ldots) = x_1^k + x_2^k + \cdots\ ,$$

and for a partition λ

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\ell(\lambda)}.$$

For a partition $\lambda \vdash n$, a semi-standard Young tableau T of shape λ is a filling of the boxes of the Young diagram of λ with positive integers in such a way that the numbers in each column increase strictly when read from top to bottom and the entries in each row weakly increase when read from left to right. The type of a semi-standard Young tableau is the sequence: $\mathrm{type}(T) = (t_1, t_2, \ldots)$ where t_i is the number of entries in T that are equal to i. Then we define the *weight* of a semi-standard Young tableau $w(T)$ to be the monomial: $w(T) = x_1^{t_1} x_2^{t_2} \cdots$.

Using the above notation one can define the Schur function combinatorially as follows:

$$s_\lambda(x_1, x_2, \ldots) = \sum_T w(T),$$

where the sum is over all semi-standard Young tableaux, T, of shape λ. It is well-known, see [14], Ch. I, (7.8), that for any partition $\mu \vdash n$

$$p_\mu = \sum_{\lambda \vdash n} \chi^\lambda_\mu s_\lambda,$$

where χ^λ_μ is the irreducible character of the symmetric group S_n evaluated at the conjugacy class indexed by μ.

1.1. **λ-Ring Notation.** Let A be a set of formal commuting variables and A^* the set of words in A. The empty word will be identified with 1. Let $c \in \mathbb{C}$, $\gamma = (\gamma_1, \ldots, \gamma_\ell) \vdash n$, $x = a_1 a_2 \ldots a_i$ be any word in A^*, and $X_1, X_2, \ldots$ be any sequence of formal sums of the words in A^* with complex coefficients. Define λ-ring notation on the power symmetric functions by

$$\begin{array}{ll} p_r[0] = 0, & p_r[1] = 1. \\ p_r[x] = x^r = a_1^r \cdots a_i^r, & p_r[cX_1] = c\, p_r[X_1]. \\ p_r[\sum_i X_i] = \sum_i p_r[X_i], & p_\gamma[X] = p_{\gamma_1}[X] \cdots p_{\gamma_\ell}[X]. \end{array}$$

where r is a nonnegative integer. These definitions imply that $p_r[X_1X_2] = p_r[X_1]p_r[X_2]$ and therefore $p_\gamma[X_1]p_\gamma[X_2]$. And for any complex number c and $\gamma \vdash n$, $p_\gamma[cX] = c^{\ell(\gamma)} p_\gamma[X]$. When $X = x_1 + x_2 + \cdots$ then

$$p_k[X] = \sum_{i \geq 1} x_i^k$$

is the usual power symmetric function. And for any partition λ, $p_\lambda[X] = p_\lambda(x_1, x_2, \ldots)$. Recall that the power symmetric functions form a basis for the ring of symmetric functions. In this notation the Schur functions can be described by

$$s_\lambda[X] = \sum_{\mu \vdash n} \frac{\chi^\lambda_\mu}{z_\mu} p_\mu[X],$$

where χ^λ_μ is the irreducible character of S_n indexed by λ evaluated at the conjugacy class indexed by μ. And $z_\mu := 1^{m_1} 2^{m_2} \cdots n^{m_n} m_1! m_2! \cdots m_n!$, where m_i is the number of parts in μ of size i.

Remark: The definition given in this section for λ-ring notation, see [20] for details, has been extended to allow for factoring out complex numbers from the function p_r. This notation allows us to derive the formulas in Section 1.3.

1.2. **Complex Reflection Groups.** Let $(\mathbb{Z}/r\mathbb{Z})$ be the cyclic group isomorphic to $\langle \epsilon \rangle$, where $\epsilon = e^{2\pi i/r}$. Then $G_{n,r} = (\mathbb{Z}/r\mathbb{Z}) \wr S_n$ is the wreath product of $(\mathbb{Z}/r\mathbb{Z})$ with the symmetric group, S_n, of degree n. Then $G_{n,r}$ is a unitary reflection group and can be identified with the group of all the permutation matrices of size n whose non-zero entries are r-th roots of unity. If $r = 1$, then $G_{n,r}$ is isomorphic to S_n and if $r = 2$, then $G_{n,r}$ is isomorphic to the the Weyl group of type B_n (the hyperoctahedral group). The order of $G_{n,r}$ is $r^n n!$. $G_{n,r}$ has the following well-known presentation:

Proposition 1.1. *The group $G_{n,r}$ is generated by $t, s_1, s_2, \ldots, s_{n-1}$ that satisfy the following relations:*

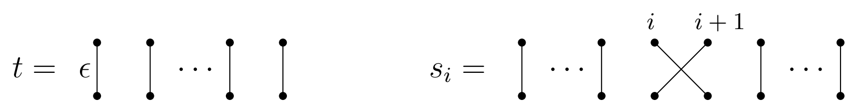
(1) $t^r = 1$;

(2) $s_i^2 = 1$ *for* $1 \leq i \leq n-1$;
(3) $ts_1ts_1 = s_1ts_1t$;
(4) $s_is_j = s_js_i$ *if* $|i-j| > 1$;
(5) $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$, *if* $1 \leq i \leq n-2$.

Any element $g \in G_{n,r}$ can be uniquely represented as the product of a permutation matrix w of size n and a diagonal matrix $\operatorname{diag}(\epsilon^{a_1}, \epsilon^{a_2}, \ldots, \epsilon^{a_n})$:

$$g = w \ \operatorname{diag}(\epsilon^{a_1}, \epsilon^{a_2}, \ldots, \epsilon^{a_n}).$$

The elements in $G_{n,r}$ can also be represented by braid-like diagrams such that the strands are labelled with elements in $\mathbb{Z}/r\mathbb{Z}$, if the label is one we omit it. For example, the generators in Proposition 1.1 have the following labelled diagrams:



where the ϵ is a label on the the first strand of t.

The product $\sigma\tau \in G_{n,r}$ can be obtained by placing τ above σ, connecting the strands and multiplying corresponding labels on the strands.

The irreducible representations of $G_{n,r}$ are indexed by r-tuples of Young diagrams $\vec{\gamma} = (\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(r)})$ of total size n. The irreducible representations can be constructed as follows, see [21] for details. Let ρ^λ denote the irreducible representation of S_n indexed by $\lambda \vdash n$. And let ϕ_i denote the irreducible representation of $\mathbb{Z}/r\mathbb{Z}$ defined by $\phi_i(\epsilon) = \epsilon^i$. Suppose that $k_1, k_2, \ldots, k_r \geq 0$ are integers such that $n = k_1 + k_2 + \ldots + k_r$ and let $\gamma^{(i)} \vdash k_i$. Then the irreducible representation of $G_{n,r}$ corresponding to $\vec{\gamma}$ is obtained by inducing the irreducible representation $\rho^{\gamma^{(1)}}\phi_0 \times \rho^{\gamma^{(2)}}\phi_1 \times \cdots \times \rho^{\gamma^{(r)}}\phi_{r-1}$ of the Young type subgroup $G_{k_1,r} \times G_{k_2,r} \times \cdots \times G_{k_r,r}$. Here $\rho^{\gamma^{(i)}}\phi_{i-1}$ is the irreducible representation of $G_{k_i,r}$ obtained as follows: Let $g = w \ \operatorname{diag}(\epsilon^{u_1}, \epsilon^{u_2}, \ldots, \epsilon^{u_k}) \in G_{k,r}$ where $w \in S_{k_i}$. Then the representing matrix $\rho^{\gamma^{(i)}}\phi_{i-1}(g)$ is given by

$$\rho^{\gamma^{(i)}} \ \phi_i(g) = \phi_i(\epsilon^{\sum_{j=1}^{k} u_j})\rho^{\gamma^{(i)}}(w).$$

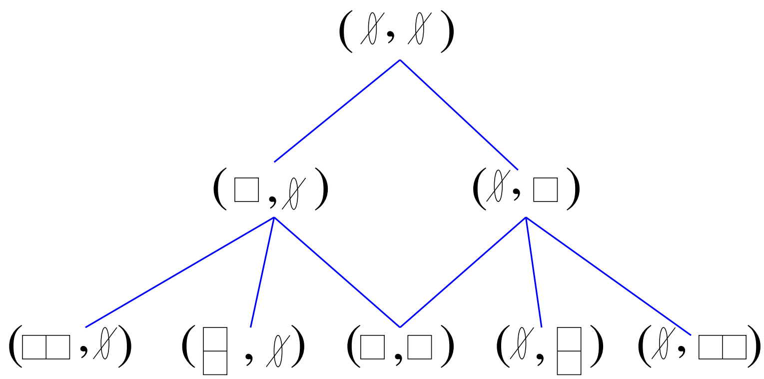
Let $f^{\vec{\gamma}}$ denote the degree of the irreducible representation parametrized by $\vec{\gamma}$. Then

$$f^{\vec{\gamma}} = \binom{n}{|\gamma^{(1)}|, \ldots, |\gamma^{(r)}|} f^{\gamma^{(1)}} f^{\gamma^{(2)}} \cdots f^{\gamma^{(r)}},$$

where $f^{\gamma^{(i)}}$ is the degree of the irreducible representation of S_n indexed by $\gamma^{(i)}$. Combinatorially, this is the number of standard Young tableaux of shape $\gamma^{(i)}$.

We have the following inclusion of groups, $G_{0,r} \subset G_{1,r} \subset G_{2,r} \subset G_{3,r} \subset \cdots$. Recall that the *Bratteli diagram* is a graph that encodes the inclusion of two algebras $A \subset B$. The vertices of the graph are labelled by an indexing set of the simple summands of

A and B. The number of edges joining a vertex v for A to a vertex w for B is the number of times that the representation indexed by v occurs in the restriction to A of the representation index by w. The following is the Bratteli diagram for the inclusion $G_{0,r} \subset G_{1,r} \subset G_{2,r}$.



1.3. **Frobenius Characteristic Map.** For details on the content of this section and the proofs and derivations of the formulas see [14] Ch. I, Appendix B, or [20]. Let $\mathcal{R}(G_{n,r})$ denote the center of the group algebra of $G_{n,r}$, that is, $\mathcal{R}(G_{n,r})$ is the set of complex valued functions on $G_{n,r}$ that are constant on conjugacy classes. Denote by $1_{\vec{\gamma}}$ the function in $\mathcal{R}(G_{n,r})$ such that if $c_{\vec{\gamma}}$ denotes the conjugacy class indexed by $\vec{\gamma}$, then

$$1_{\vec{\gamma}}(g) = \begin{cases} 1 & \text{if } g \in c_{\vec{\gamma}} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $\{1_{\vec{\gamma}} \mid \vec{\gamma} \vdash n\}$ is a basis for $\mathcal{R}(G_{n,r})$. Recall that the inner product for $f, g \in \mathcal{R}(G_{n,r})$ is defined as follows:

$$\langle f, g \rangle_{G_{n,r}} := \frac{1}{r^n n!} \sum_{\sigma \in G_{n,r}} f(\sigma)\overline{g(\sigma)}.$$

Under this inner product we have

$$\langle 1_{\vec{\gamma}}, 1_{\vec{\nu}} \rangle_{G_{n,r}} = \delta_{\vec{\gamma}\vec{\nu}} \prod_{i+1}^{r} \frac{1}{z^{\gamma^{(i)}}} (1/r)^{\ell(\gamma^{(i)})}.$$

For $i = 1, \ldots, r$ and variables $x_1^i, x_2^i, \ldots$. Let $X^i = x_1^i + x_2^i + \cdots$. Define

$$\Lambda_n(G_{n,r}) := \bigoplus_{n_1 + \cdots + n_r = n} \bigotimes_{i=1}^{r} \Lambda_{n_i}(X^i).$$

where $\Lambda_{n_i}(X)$ denotes the ring of symmetric functions of degree n_i in the variables X. For any basis $\{a_\lambda \,|\, \lambda \vdash n\}$ of $\lambda_n(X^i)$, we get that

$$\left\{ \prod_{i=1}^{r} a_{\gamma^{(i)}}[X^i] \;:\; \vec{\gamma} \vdash n \right\}$$

is a basis for $\Lambda_n(G_{n,r})$.

The Frobenius characteristic map, $ch : \mathcal{R}(G_{n,r}) \to \Lambda_n(G_{n,r})$ is the isomophism defined by

$$ch(1_{\vec{\gamma}}) = \prod_{i=1}^{r} \frac{p_{\gamma^{(i)}}[X^i]}{z_{\gamma^{(i)}}}.$$

Set $P_{t(\sigma)} := \prod_{i=1}^{r} p_{\gamma^{(i)}}[X^i] r^{\ell(\gamma^{(i)})}$, where $t(\sigma)$ denotes the conjugacy type of σ, in this case $t(\sigma) = \vec{\gamma}$. For any $f \in \mathcal{R}(G_{n,r})$ we have

$$ch(f) := \frac{1}{r^n n!} \sum_{\sigma \in G_{n,r}} f(\sigma) P_{t(\sigma)}.$$

Using λ-ring notation one can derive the following proposition, for the details of the proof see [20], Corollary 13. The proposition has been proved in greater generality for any wreath product of any finite group G with S_n in [14], Ch. I, Appendix B, (9.4).

Proposition 1.2. *For $\vec{\gamma} \vdash n$,*

$$ch(\chi^{\vec{\gamma}}) = \prod_{i=1}^{r} s_{\gamma^{(i)}}[\sum_{j=1}^{r} \phi_j^i X^j].$$

where ϕ_j^i is the character of $\mathbb{Z}/r\mathbb{Z}$ indexed by j evaluated at ϵ^i.

1.4. **A permutation representation of $G_{n,r}$.** There exists an action of $G_{n,r}$ on the set $\{1, 2, \ldots, n\} \times \mathbb{Z}/r\mathbb{Z}$ defined by

$$g \cdot (i, \epsilon^j) = (\sigma(i), \epsilon^{u_i + j}), \qquad \text{where } g = \sigma \ \mathrm{diag}(\epsilon^{u_1}, \ldots, \epsilon^{u_r}).$$

It is straightforward to show that this action is well-defined. Using this action we define a $G_{n,r}$-module. Let $W \cong \mathbb{C}\text{-Span}\{v_{(i,\epsilon^j)} \;:\; 1 \le i \le n, 0 \le j \le r-1\}$ where for any $g \in G_{n,r}$,

$$g \cdot v_{(i,\epsilon^j)} = v_{g\cdot(i,\epsilon^j)}. \tag{1}$$

We remark that the representation W is isomorphic to the permutation representation of $G_{n,r}$ with respect to $G_{n-1,r}$. That is, $\mathbf{1} \uparrow_{G_{n-1,r}}^{G_{n,r}}$, where $\mathbf{1}$ denotes the trivial representation of $G_{n-1,r}$.

Remarks: (1) The dimension of W is nr.

(2) If $r = 1$, then $G_{n,r}$ is isomorphic to the symmetric group and in this case W is the permutation representation of S_n. It is well-known that in this case $W = V_{(n)} \oplus V_{(n-1,1)}$, where for any partition $\lambda \vdash n$, V_λ denotes the corresponding irreducible representation.

(3) Throughout this paper we will use the indexing of the irreducible representations of $G_{n,r}$ used by Ariki-Koike [1] Corollary 3.14. In particular, the representations indexed by $(\lambda, \emptyset, \dots, \emptyset)$ correspond to the representations of the symmetric group S_n in a natural way. That is, the generator t acts trivially on all these modules.

In the next proposition we decompose the $G_{n,r}$-module W in terms of its irreducible constituents. We will use the next proposition in Section 2 to decompose $W^{\otimes k}$.

Proposition 1.3. *As a $G_{n,r}$-module the decomposition of W is*

$$W \cong V_{((n),\emptyset,\dots,\emptyset)} \oplus V_{((n-1,1),\emptyset,\dots,\emptyset)} \oplus \bigoplus_{k=2}^{r} V_{((n-1),\emptyset,\dots,\emptyset,\underbrace{(1)}_{k},\emptyset,\dots,\emptyset)}. \tag{2}$$

Proof. We will construct r subspaces and show that under the action described in Equation (1) they correspond to the representations in the right hand side of (2).

For $0 \le i \le r-1$ define the subspace W^i as the linear span of the following vectors:

$$w_j^i := \epsilon^{i\cdot r} v_{(j,\epsilon^0)} + \epsilon^{i(r-1)} v_{(j,\epsilon^1)} + \cdots + \epsilon^{i\cdot 2} v_{(j,\epsilon^{r-2})} + \epsilon^{i\cdot 1} v_{(j,\epsilon^{r-1})}, \qquad \text{for } j = 1, \dots, n.$$

It is an easy exercise in linear algebra to check that as vector spaces $W \cong \oplus_{i=0}^{r-1} W^i$. Observe that

$$t.w_1^i = \epsilon^i\, w_1^i \qquad \text{and} \qquad t.w_j^i = w_j^i, \quad \text{for } j > 1,$$

and for any permutation $\sigma \in S_n$

$$\sigma.w_j^i = w_{\sigma(j)}^i.$$

According to Corollary 3.14 in [1], if $i > 0$, then $W^i \cong V_{((n-1),\emptyset,\dots,\emptyset,\underbrace{(1)}_{i+1},\emptyset,\dots,\emptyset)}$. And if $i = 0$, then t acts trivially on W^0, hence W^0 is isomorphic to the permutation representation of the symmetric group. It is well-known that $W^0 \cong V_{((n),\emptyset,\dots,\emptyset)} \oplus V_{((n-1,1),\emptyset,\dots,\emptyset)}$. Thus, the claim follows. $\square$

2. Kronecker products of characters of $G_{n,r}$

Let $f_1, f_2 \in \mathcal{R}(G_{n,r})$ define the *Kronecker product* $f_1 \otimes f_2$ by $(f_1 \otimes f_2)(g) = f_1(g)f_2(g)$. Hence $f_1 \otimes f_2 \in \mathcal{R}(G_{n,r})$. Given finite-dimensional representations of $G_{n,r}$, $\rho_1 : G_{n,r} \to GL(V_1)$ and $\rho_2 : G_{n,r} \to GL(V_2)$, then define the tensor product representation $\rho_1 \otimes \rho_2 : G_{n,r} \to GL(V_1 \otimes V_2)$ via the diagonal action:

$$g.(x \otimes y) = g.x \otimes g.y.$$

Then the character of $\rho_1 \otimes \rho_2$ is known as the *Kronecker product* of the characters χ^{ρ_1} and χ^{ρ_2}, i.e. the pointwise product $\chi^{\rho_1} \otimes \chi^{\rho_2}(w) = \chi^{\rho^1}(w) \otimes \chi^{\rho^2}(w)$, for $w \in G_{n,r}$.

For $\vec{\gamma} \vdash n$ we will set $s_{\vec{\gamma}} := \prod_{i=1}^{r} s_{\gamma^{(i)}}[\sum_{j=1}^{r} \epsilon^{(i-1)j} X^j]$. In this notation we have that

$$ch(\chi^{\vec{\gamma}}) = s_{\vec{\gamma}}.$$

With this notation we have that $s_{\gamma^{(i)}}[\sum_{j=1}^{r} \epsilon^{(i-1)j} X^j] = s_{(\emptyset,\ldots,\emptyset,\gamma^{(i)},\emptyset,\ldots,\emptyset)}$. Hence,

$$s_{\vec{\gamma}} = s_{(\gamma^{(1)},\emptyset,\ldots,\emptyset)} s_{(\emptyset,\gamma^{(2)},\emptyset,\ldots\emptyset)} \cdots s_{(\emptyset,\ldots,\emptyset,\gamma^{(r)})}.$$

One can easily see that for $\vec{\gamma} \vdash n$ and $\vec{\delta} \vdash m$ we have

$$s_{\vec{\gamma}} s_{\vec{\delta}} = \sum_{\vec{\nu} \vdash n+m} \prod_{i=1}^{r} c^{\nu^{(i)}}_{\gamma^{(i)},\delta^{(i)}} s_{(\emptyset,\ldots,\emptyset,\nu^{(i)},\emptyset,\ldots,\emptyset)},$$

where $c^{\nu^{(i)}}_{\gamma^{(i)},\delta^{(i)}}$ are the Littlewood-Richardson coefficients. For simplicity, we let $\chi^{\vec{\gamma}}$ denote the character of the irreducible representation of $G_{n,r}$ indexed by $\vec{\gamma}$ and $\chi^{\vec{\gamma}\times\vec{\delta}}$ denote the corresponding irreducible character of $G_{n,r} \times G_{m,r}$. Using the Frobenius characteristic map we have that $ch(\chi^{\vec{\gamma}\times\vec{\delta}\uparrow^{G_{n+m,r}}_{G_{n,r}\times G_{m,r}}}) = s_{\vec{\gamma}} s_{\vec{\delta}}$. Then in terms of the inner product we have

$$\langle \chi^{\vec{\nu}}, \chi^{\vec{\gamma}\times\vec{\delta}\uparrow^{G_{n+m,r}}_{G_{n,r}\times G_{m,r}}} \rangle = \prod_{i=1}^{r} c^{\nu^{(i)}}_{\gamma^{(i)},\delta^{(i)}}.$$

We set $C^{\vec{\gamma}}_{\vec{\gamma},\vec{\delta}} := \prod_{i=1}^{r} c^{\nu^{(i)}}_{\gamma^{(i)},\delta^{(i)}}$ to simplify the expressions. By Frobenius reciprocity we also have that

$$\chi^{\vec{\nu}} \downarrow^{G_{n+m,r}}_{G_{n,r}\times G_{m,r}} = \sum_{\substack{\vec{\gamma}\vdash n \\ \vec{\delta}\vdash m}} C^{\vec{\nu}}_{\vec{\gamma},\vec{\delta}} \chi^{\vec{\gamma}} \chi^{\vec{\delta}}. \tag{3}$$

We can now define the Kronecker product of Schur functions by

$$s_{\vec{\gamma}} \otimes s_{\vec{\delta}} := ch(\chi^{\vec{\gamma}} \otimes \chi^{\vec{\delta}}).$$

The trivial representation of $G_{n,r}$ is indexed by $\chi^{((n),\emptyset,\ldots,\emptyset)}$. Hence $\chi^{\vec{\nu}} \otimes \chi^{((n),\emptyset,\ldots,\emptyset)} = \chi^{\vec{\nu}}$. The proof of the following theorem is in [20], it is a combinatorial proof that uses the notion of $\star$-rim hook tableaux.

Theorem 2.1 ([20], Th. 24). *Let $\nu = (\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(r)})$, then for $k \geq 2$*

$$\chi^{\vec{\nu}} \otimes \chi^{(\emptyset,\ldots,\emptyset,(n)_k,\emptyset,\ldots,\emptyset)} = \chi^{(\nu^{(r-i+2)},\ldots,\nu^{(r)},\nu^{(1)},\ldots,\nu^{(r-i+1)})}$$

where $(\emptyset,\ldots,\emptyset,(n)_k,\emptyset,\ldots,\emptyset)$ denotes that the partition (n) occurs in the k-th coordinate.

Garsia-Remmel [5] (6.15) proved that

$$s_\lambda s_\mu \otimes s_\nu = \sum_{\substack{\alpha\vdash n \\ \alpha\subseteq\nu}} (s_\lambda \otimes s_\alpha)(s_\mu \otimes s_{\nu/\alpha}).$$

The following theorem generalizes this result to Schur functions corresponding to the characters of complex reflection groups. We will use the formula in the theorem to decompose more general tensor products that we will need for the partition algebras.

Theorem 2.2. *Let $\vec{\gamma} \vdash n$, $\vec{\delta} \vdash m$. Then*

$$s_{\vec{\gamma}} s_{\vec{\delta}} \otimes s_{\vec{\nu}} = \sum_{\substack{\vec{\mu} \vdash n \\ \vec{\mu} \subseteq \vec{\nu}}} (s_{\vec{\gamma}} \otimes s_{\vec{\mu}})(s_{\vec{\delta}} \otimes s_{\vec{\nu}/\vec{\mu}})$$

Proof. Using the Frobenius characteristic map from Section 1.3 we have

$$\begin{aligned} s_{\vec{\gamma}} s_{\vec{\delta}} \otimes s_{\nu} &= ch(\chi^{\vec{\gamma} \times \vec{\delta} \uparrow_{G_{n,r} \times G_{m,r}}^{G_{n+m,r}}} \otimes \chi^{\vec{\nu}}) \\ &= \frac{1}{|G_{n+m,r}|} \sum_{\phi \in G_{n+m,r}} \chi^{\vec{\gamma} \times \vec{\delta} \uparrow_{G_{n,r} \times G_{m,r}}^{G_{n+m,r}}} (\phi) \otimes \chi^{\vec{\nu}}(\phi) P_{t(\phi)} \end{aligned}$$

Now we apply the definition of induced representation and simplify

$$\begin{aligned} s_{\vec{\gamma}} s_{\vec{\delta}} \otimes s_{\nu} &= \frac{1}{|G_{n,r}||G_{m,r}|} \sum_{\substack{\sigma \in G_{n,r} \\ \tau \in G_{m,r}}} \chi^{\vec{\gamma}}(\sigma) \chi^{\vec{\delta}}(\tau) \chi^{\vec{\nu}}(\gamma, \tau) P_{t(\gamma,\tau)} \\ &= \frac{1}{|G_{n,r}||G_{m,r}|} \sum_{\substack{\sigma \in G_{n,r} \\ \tau \in G_{m,r}}} \chi^{\vec{\gamma}}(\sigma) \chi^{\vec{\delta}}(\tau) \sum_{\substack{\vec{\gamma} \vdash n \\ \vec{\delta} \vdash m}} C^{\vec{\nu}}_{\vec{\xi},\vec{\eta}} \chi^{\vec{\xi}}(\sigma) \chi^{\vec{\eta}}(\tau) P_{t(\sigma)} P_{t(\tau)} \end{aligned}$$

The last equality follows from the fact that $\chi^{\vec{\gamma}}(\sigma, \tau)$ is restricted to $G_{n,r} \times G_{m,r}$ and equation (3). We now regroup the terms and we get that the right-handside of our last equation can be rewritten as

$$\sum_{\substack{\vec{\gamma} \vdash n \\ \vec{\delta} \vdash m}} C^{\vec{\nu}}_{\vec{\xi},\vec{\eta}} \left(\sum_{\gamma \in G_{n,r}} \frac{\chi^{\vec{\gamma}}(\sigma) \chi^{\vec{\xi}}(\sigma) P_{t(\sigma)}}{|G_{n,r}|} \right) \left(\sum_{\tau \in G_{m,r}} \frac{\chi^{\vec{\delta}}(\tau) \chi^{\vec{\eta}}(\tau) P_{t(\tau)}}{|G_{m,r}|} \right) = \sum_{\substack{\vec{\xi} \vdash n \\ \vec{\eta} \vdash m}} C^{\vec{\nu}}_{\vec{\xi},\vec{\eta}} (s_{\vec{\gamma}} \otimes s_{\vec{\xi}})(s_{\vec{\delta}} \otimes s_{\vec{\eta}})$$

Since $s_{\vec{\nu}/\vec{\xi}} = \sum_{\vec{\eta} \vdash m} C^{\vec{\nu}}_{\vec{\xi},\vec{\eta}} s_{\vec{\eta}}$ then by substituting in the last expression we obtain the right-handside in the last equation is equal to $\sum_{\substack{\vec{\xi} \vdash n \\ \vec{\xi} \subseteq \vec{\nu}}} (s_{\vec{\gamma}} \otimes s_{\vec{\xi}})(s_{\vec{\delta}} \otimes s_{\vec{\nu}/\vec{\xi}})$. □

Theorem 2.3. *For $i = 2, \ldots, r$, let $((n-1), \emptyset, \ldots, \emptyset, (1)_i, \emptyset, \ldots, \emptyset)$ denote the r-tuple with partition (1) in the ith coordinate and $(n-1)$ in the first coordinate. Then for $\vec{\nu} \vdash n$,*

$$s_{((n-1),\emptyset,\ldots,\emptyset,(1)_i,\emptyset,\ldots,\emptyset)} \otimes s_{\vec{\nu}} = \sum_{\vec{\mu}} s_{\vec{\mu}}$$

where $\vec{\mu}$ runs over all $\vec{\mu} \vdash n$ that are obtained by removing a box from $\vec{\nu}$, then adding a box in such a way that if the box removed was in the k-th position we add to the $i + k - 1$-th modulo r coordinate.

Proof. By Theorem 2.2

$$s_{((n-1),\emptyset,\dots,\emptyset)}s_{(\emptyset,\dots,\emptyset,(1)_i,\emptyset,\dots,\emptyset)} \otimes s_{\vec{\nu}} = \sum_{\substack{\vec{\xi}\vdash n-1\\ \vec{\xi}\subseteq\vec{\nu}}} \big(s_{((n-1),\emptyset,\dots,\emptyset)} \otimes s_{\vec{\xi}}\big)\big(s_{(\emptyset,\dots,\emptyset,(1)_i,\emptyset,\dots,\emptyset)} \otimes s_{\vec{\nu}/\vec{\xi}}\big). \tag{4}$$

Notice that $s_{((n-1),\emptyset,\dots,\emptyset)} \otimes s_{\vec{\xi}} = s_{\vec{\xi}}$ and $s_{\vec{\nu}/\vec{\xi}} = s_{(\emptyset,\dots,\emptyset,(1)_k,\emptyset,\dots,\emptyset)}$ for some $k = 1,\dots,r$. Hence, $s_{(\emptyset,\dots,\emptyset,(1)_i,\emptyset,\dots,\emptyset)} \otimes s_{\vec{\nu}/\vec{\xi}} = s_{(\emptyset,\dots,\emptyset,(1)_i,\emptyset,\dots,\emptyset)} \otimes s_{(\emptyset,\dots,\emptyset,(1)_k,\emptyset,\dots,\emptyset)}$. By Theorem 2.1 we have $s_{(\emptyset,\dots,\emptyset,(1)_i,\emptyset,\dots,\emptyset)} \otimes s_{(\emptyset,\dots,\emptyset,(1)_k,\emptyset,\dots,\emptyset)} = s_{(\emptyset,\dots,\emptyset,(1)_j,\emptyset,\dots,\emptyset)}$ where $1 \le j \le r-1$ and $k-1+i \equiv j (\mathrm{mod}\ r)$. This implies that the right hand side of Equation 4 is equal to

$$\sum_{\substack{\vec{\xi}\vdash n-1\\ \vec{\xi}\subseteq\vec{\nu}}} s_{\vec{\xi}}s_{(\emptyset,\dots,\emptyset,(1)_j,\emptyset,\dots,\emptyset)}.$$

This last expression is equivalent to our claim. $\square$

Example. Let $r = 3$ and $\vec{\gamma} = \left(\ydiagram{2,1}, \ydiagram{1}, \ydiagram{2}\right)$. And let $U = V_{((5),(1),\emptyset)}$. Then,

$$\begin{aligned} U\otimes V_{\vec{\gamma}} = {} & V_{\left(\ydiagram{1,1}, \ydiagram{2}, \ydiagram{2}\right)} \oplus V_{\left(\ydiagram{1,1}, \ydiagram{1,1}, \ydiagram{2}\right)} \oplus V_{\left(\ydiagram{2}, \ydiagram{2}, \ydiagram{2}\right)} \oplus V_{\left(\ydiagram{2}, \ydiagram{1,1}, \ydiagram{2}\right)} \oplus \\ & V_{\left(\ydiagram{2,1}, \emptyset, \ydiagram{3}\right)} \oplus V_{\left(\ydiagram{2,1}, \emptyset, \ydiagram{2,1}\right)} \oplus V_{\left(\ydiagram{3,1}, \ydiagram{1}, \ydiagram{1}\right)} \oplus V_{\left(\ydiagram{2,2}, \ydiagram{1}, \ydiagram{1}\right)} \oplus \\ & V_{\left(\ydiagram{2,1,1}, \ydiagram{1}, \ydiagram{1}\right)}. \end{aligned}$$

Proposition 2.4. *Let $\vec{\gamma} \vdash n$, then*

$$\big(s_{((n),\emptyset,\dots,\emptyset)} + s_{((n-1,1),\emptyset,\dots,\emptyset)}\big) \otimes s_{\vec{\gamma}} = \sum_{\vec{\mu}} s_{\vec{\mu}}$$

where the sum runs over all $\vec{\mu}$ that are obtained from $\vec{\gamma}$ by first removing a box from the k- coordinate of $\vec{\gamma}$ and then adding a box to the Young diagram in the k-th coordinate, where $1 \le k \le r$.

Proof. The proof of this proposition is similar to the proof of 2.3. Notice that

$$s_{((n-1),\emptyset,\dots,\emptyset)}s_{((1),\emptyset,\dots,\emptyset)} = s_{((n),\emptyset,\dots\emptyset)} + s_{((n-1,1),\emptyset,\dots,\emptyset)}.$$

Now by Theorem 2.2

$$s_{((n-1),\emptyset,\dots,\emptyset)}s_{(\emptyset,\dots,\emptyset,(1)_i,\emptyset,\dots,\emptyset)} \otimes s_{\vec{\nu}} = \sum_{\substack{\vec{\xi}\vdash n-1\\ \vec{\xi}\subseteq\vec{\nu}}} \big(s_{((n-1),\emptyset,\dots,\emptyset)} \otimes s_{\vec{\xi}}\big)\big(s_{((1),\emptyset,\dots,\emptyset)} \otimes s_{\vec{\nu}/\vec{\xi}}\big).$$

Suppose that $\vec{\nu}/\vec{\xi} = (\emptyset, \dots, \emptyset, (1)_k, \emptyset, \dots, \emptyset)$ for some $1 \leq k \leq r$, then $s_{((1),\emptyset,\dots,\emptyset)} \otimes s_{\vec{\nu}/\vec{\xi}} = s_{(\emptyset,\dots,\emptyset,(1)_k,\emptyset,\dots,\emptyset)}$. Hence the right-handside of the previous equation can be simplified to

$$\sum_{\substack{\vec{\xi} \vdash n-1 \\ \vec{\xi} \subseteq \vec{\nu}}} s_{\vec{\xi}} s_{(\emptyset,\dots,\emptyset,(1)_k,\emptyset,\dots,\emptyset)}.$$

Hence, the last expression can be interpreted as first removing and then adding a box from the k-th coordinate as claimed in the statement of the theorem. □

Corollary 2.5. *Let W be the permutation module defined in (1) and denote by $V_{\vec{\gamma}}$ the irreducible $G_{n,r}$-module indexed by $\vec{\gamma}$. Then*

$$W \otimes V_{\vec{\gamma}} \cong \bigoplus_{\vec{\delta}} V_{\vec{\delta}},$$

where the sum runs over all r-tuples $\vec{\delta} \vdash n$ that are obtained by removing a box from any possible component of $\vec{\gamma}$ to obtained an r-tuple $\vec{\xi} \vdash n-1$ and then adding a box to $\vec{\xi}$ in any possible component to obtain $\vec{\delta} \vdash n$.

Proof. This is a direct consequence of 2.3 and Proposition 2.4. □

Example. Let $r = 2$ and $\vec{\gamma} = \left(\yng(2,1), \yng(2)\right)$. Then,

$$\begin{aligned} W \otimes V_{\vec{\gamma}} = \; & V_{\left(\yng(2), \yng(3)\right)} \oplus V_{\left(\yng(2), \yng(2,1)\right)} \oplus V_{\left(\yng(1,1), \yng(3)\right)} \oplus V_{\left(\yng(1,1), \yng(2,1)\right)} \oplus V_{\left(\yng(3,1), \yng(1)\right)} \oplus \\ & V_{\left(\yng(2,2), \yng(1)\right)} \oplus V_{\left(\yng(2,1,1), \yng(1)\right)} \oplus 3V_{\left(\yng(2,1), \yng(2)\right)} \oplus V_{\left(\yng(1,1,1), \yng(2)\right)} \oplus V_{\left(\yng(3), \yng(2)\right)} \oplus \\ & V_{\left(\yng(2,1), \yng(1,1)\right)}. \end{aligned}$$

Note that this product is not multiplicity free.

3. Partition algebras of the symmetric group

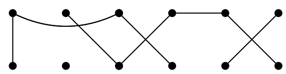
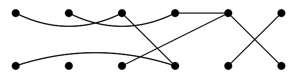
In this section we recall some of the known results about the partition algebra that will be used when we show similar results for the partition algebras related to the complex reflection groups. The results in this section can be found in [9, 15, 16, 17, 18], the reader interested in more details should consult these papers. The exposition given here follows that of [9].

Let k be a positive integer and define

$$A_k := \{\text{set partitions of } \{1, 2, \dots, k, 1', \dots, k'\}\}.$$

The set partitions $d \in A_k$ are represented by a graph with k vertices in the top row labelled $1, 2, \dots, k$ from left to right, and k vertices on the bottom row, labelled $1', 2', \dots, k'$

from left to right. In some cases we will identify i' with $i+k$ to simplify some of the exposition. In this diagram a vertex i and a vertex j are connected by a path if i and j are in the same block of the set partition. The graph representing a set partition is not unique. We say that two graphs are equivalent if they give rise to the same set partition. The term *partition diagram* will be used to mean the equivalence class of the given graph. For example,

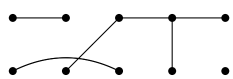
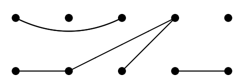
 $\equiv$ 

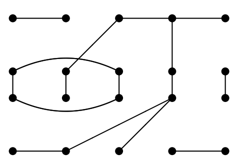
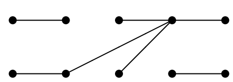
The composition $d_2 \circ d_1$ of partition diagrams, $d_1, d_2 \in A_k$ is the set partition obtained by placing d_1 above d_2 and identifying the bottom vertices of d_1 with the top vertices of d_2, removing any connected components that contain only vertices in the middle row.

Let x be an indeterminate. The *partition algebra* $\mathcal{P}_k(x)$ is defined as the associative algebra over $\mathbb{C}(x)$ with basis A_k,

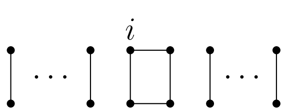
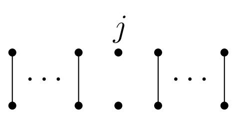
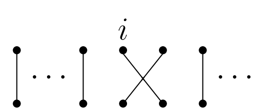
$$\mathcal{P}_k(n) := \mathbb{C}(x)\text{-span } \{d \in A_k\},$$

with multiplication defined by $d_2 d_1 = n^{\gamma}(d_2 \circ d_1)$, where γ is the number of blocks removed from the middle row when constructing the composition $d_1 \circ d_2$. For example, if

$d_1 =$  and $d_2 =$  then

$d_2 d_1 =$  $= x^2$ 

Halverson and Ram [9] have proved a presentation for $\mathcal{P}_k(n)$. Let k be a positive integer. For $1 \le i \le k-1$ and $1 \le j \le k$, define

$p_{i+\frac{1}{2}} =$  $p_j =$  $s_i =$ 

Proposition 3.1 ([9], Theorem 1.11)**.** *The set* $\{p_{i+\frac{1}{2}}, p_j, s_i : 1 \le i \le n-1, 1 \le j \le n\}$ *together with the relations given below yield a presentation for* $\mathcal{P}_k(x)$. *Let* $a_i = 1$ *if* $i \in \frac{1}{2}\mathbb{Z}_{>0}$ *and zero otherwise.*

$$p_i^2 = x^{a_i} p_i, \qquad p_i p_{i\pm\frac{1}{2}} p_i = p_i \qquad \textit{and} \qquad p_i p_j = p_j p_i, \quad \textit{for } |i-j| > 1/2.$$

$$s_i^2 = 1, \qquad s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, \qquad \textit{and} \qquad s_is_j = s_js_i, \quad \textit{for } |i-j| > 1.$$

and

$$s_ip_ip_{i+1} = p_ip_{i+1}s_i = p_ip_{i+1}, \qquad s_ip_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}s_i = p_{i+\frac{1}{2}}, \qquad s_ip_is_i = p_{i+1},$$

$$s_is_{i+1}p_{i+\frac{1}{2}}s_{i+1}s_i = p_{i+\frac{3}{2}}, \qquad \textit{and} \qquad s_ip_j = p_js_i, \quad \textit{for } j \neq i - \frac{1}{2}, i, i+\frac{1}{2}, i+1, i+\frac{3}{2}.$$

Theorem 3.2 (Martin and Saleur [19]). *For each integer $n \geq 0$, $\mathcal{P}_k(x)$ is semisimple over $\mathbb{C}(x)$, and $\mathcal{P}_k(\xi)$ is semisimple over $\mathbb{C}$ whenever ξ is not an integer in $\{0, 1, \ldots, 2k-1\}$.*

3.1. **Schur-Weyl Duality.** Let $V = \mathbb{C}^n$ with standard basis $v_1, v_2, \ldots, v_n$ and let the permutation group S_n act on V as follows,

$$\sigma(v_i) = v_{\sigma(i)}, \qquad \text{for } \sigma \in S_n \quad \text{and } 1 \leq i \leq n. \tag{5}$$

Hence, V is the permutation module of S_n. For each positive integer k, $V^{\otimes k}$ is an S_n-module where S_n acts on $V^{\otimes k}$ via the diagonal action,

$$\sigma(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = v_{\sigma(i_1)} \otimes \cdots \otimes v_{\sigma(i_k)}.$$

For $d \in A_k$ and values $i_1, \ldots, i_k, i_{k+1}, \ldots, i_{2k} \in \{1, \ldots, n\}$ define

$$(d)^{i_1,\ldots,i_k}_{i_{k+1},\ldots,i_{2k}} = \begin{cases} 1 & \text{if } i_r = i_s \text{ when } r \text{ and } s \text{ are in the same block of } d, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Define an action of a partition diagram $d \in \mathcal{P}_k(n)$ on $V^{\otimes k}$ by defining it on the standard basis by

$$(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) \cdot d = \sum_{1 \leq i_{k+1}, \ldots, i_{2k} \leq n} (d)^{i_1,\ldots,i_k}_{i_{k+1},\ldots,i_{2k}} v_{i_1} \otimes \cdots \otimes v_{i_{2k}}.$$

Theorem 3.3 (Jones [10]). *S_n and $\mathcal{P}_k(n)$ generate full centralizers of each other in $End(V^{\otimes k})$.*

(i) *$\mathcal{P}_k(n)$ generates $End_{S_n}(V^{\otimes k})$, and when $n \geq 2k$, $\mathcal{P}_k(n) \cong End_{S_n}(V^{\otimes k})$.*
(ii) *S_n generates $End_{\mathcal{P}_k(n)}(V^{\otimes k})$.*

3.2. **The irreducible representations of $\mathcal{P}_k(n)$.** The irreducible representations of the symmetric group S_r are indexed by partitions $\lambda \vdash n$. We denote these representations by S^λ. It is well known that the permutation module V of S_n, see Equation (5), decomposes as follows:

$$V \cong S^{(n)} \oplus S^{(n-1,1)}.$$

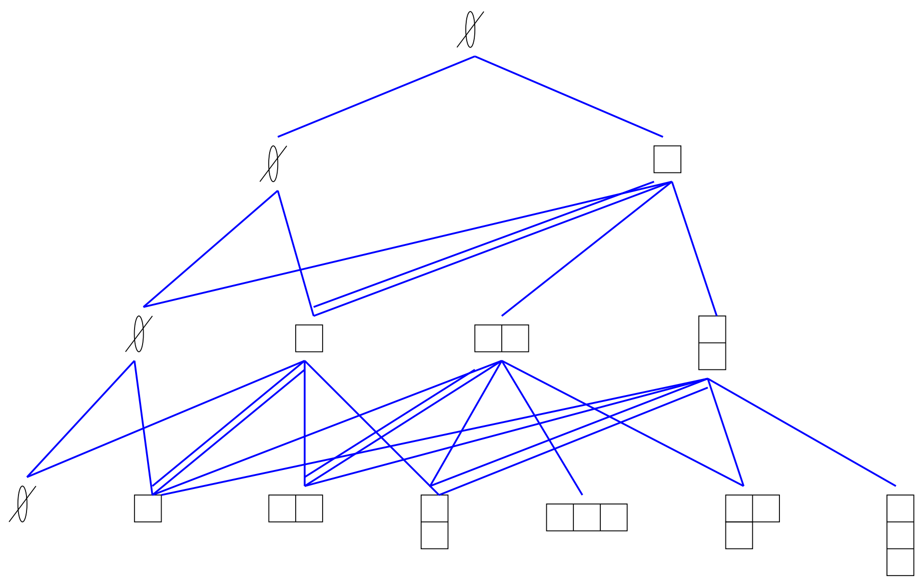
The rule for decomposing the Kronecker product of the module for V with any other S_n-module is well known [14] Ex. 23(d). Let $\lambda \vdash n$, then we have

$$V \otimes S^\lambda \cong \bigoplus_\mu S^\mu,$$

where the sum is over all $\mu \vdash n$ that can be obtained by removing one box from λ to get $\nu \vdash n-1$ and then adding a box back to ν. Using this rule we can now recursively construct the Bratteli diagram for the tower of partition algebras:

$$\mathcal{P}_0(n) \subset \mathcal{P}_1(n) \subset \mathcal{P}_2(n) \subset \mathcal{P}_3(n) \subset \cdots .$$

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ be a partition of n and $\lambda^* = (\lambda_2, \lambda_3, \ldots, \lambda_{\ell(\lambda)})$. In the Bratteli diagram below we label the vertices with partitions λ^* with the understanding that they can be completed to a partition of n by adding the first row, $\lambda_1 = n - |\lambda^*|$. In the following diagram we give the first four levels of the Bratteli diagram for $\mathcal{P}_0(n) \subset \mathcal{P}_1(n) \subset \mathcal{P}_2(n) \subset \mathcal{P}_3(n)$:



The results in the following theorem have appeared in several papers, for example [17] and [18], the formulation presented here is the one given in [9].

Theorem 3.4. *Let $n, k \in \mathbb{Z}_{\geq 0}$. Let S^λ denote the irreducible S_n-module indexed by λ. Then we have the following*

(a) *The irreducible representations of $\mathcal{P}_k(n)$ are indexed by the same set the irreducible S_n-modules in $V^{\otimes n}$. In particular this set is given by*

$$\Lambda(n,k) = \{\lambda \vdash n \,|\, |\lambda^*| \leq n\},$$

where $\lambda^* = (\lambda_2, \ldots, \lambda_{\ell(\lambda)})$.

(b) *The dimension of an irreducible representation M^λ of $\mathcal{P}_k(n)$ is equal to the multiplicity of S^λ in $V^{\otimes k}$. This number is equal to the number of paths from the top of the Bratteli diagram to λ.*

(c) *The decomposition of* $V^{\otimes k}$ *as a bimodule for* $S_n \times \mathcal{P}_k(n)$ *is*

$$V^{\otimes k} \cong \bigoplus_{\lambda \in \Lambda(n,k)} S^\lambda \otimes M^\lambda,$$

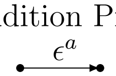
where M^λ *denotes the irreducible representations of* $\mathcal{P}_k(n)$.

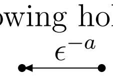
Proof. The proof of this theorem is a direct consequence of the double centralizer theory, in particular the Centralizer Theorem [7] Theorem 3.5, or [3] Sect. 3D. □

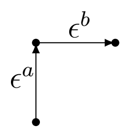
4. Colored partition algebra

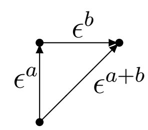
The colored partition algebra was introduced by Bloss [2] as a generalization of the partition algebra. Let C_r denote the cyclic group generated by $\epsilon = e^{2\pi i/r}$. A *colored partition diagram* is a partition graph so that the edges are oriented and each edge is labelled by the elements of C_r, and in addition these graphs are subject to the following equivalence relation:

- The underlying partition diagrams are equivalent.
- "Vector Addition Property": The following hold for edge colored diagrams:

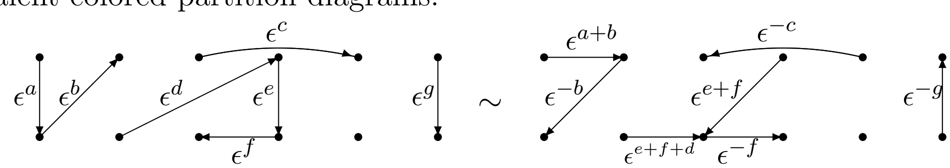


is equivalent to




is equivalent to


Here the "underlying" diagram refers to the unlabelled partition diagram with equivalence relation given by path connectedness, see Section 3. For example, the following are equivalent colored partition diagrams:



Denote by $B(k,r)$ the set of colored partition diagrams with $2k$ vertices and edges labelled by elements of C_r. Then the *colored partition algebra* is the associative algebra over $\mathbb{C}(x)$ with identity defined as

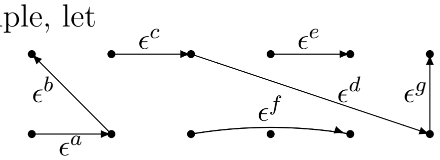
$$\mathcal{P}_k(x,r) := \mathbb{C}(x)\text{span-}\{d \ : \ d \in B(k,r)\}.$$

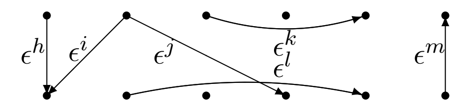
The multiplication of colored partition diagrams is as follows: Let $d_1, d_2 \in B(k,r)$, to form the product $d_2 d_1$ of d_2 and d_1

- Multiply the underlying partition diagrams d_1 and d_2, this yields the underlying partition diagram for the colored partition diagram $d_2 d_1$.
- If when we place d_1 on top of d_2 and a bottom edge of d_1 coincides with a top edge of d_2 and both edges have the same direction but different labels, then the product $d_2 d_1 = 0$.

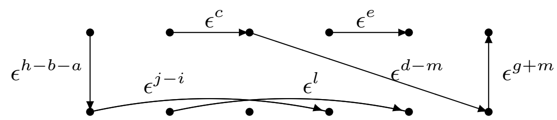
- To label the underlying diagram we use the "vector addition property" making sure this property holds for all edges of the underlying diagram.
- Let γ be the number of connected components containing only "middle vertices", then we multiply $d_2 d_1$ by $(rx)^\gamma$.

For example, let

$d_1 =$ 

$d_2 =$ 

Hence, we have

$d_2 d_1 = (\delta_{a,h-i}\delta_{f,-k}xr)$ 

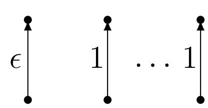
By counting colored partition diagrams we obtain the dimension of $\mathcal{P}_k(x,r)$, see Section 6 in [2] for details.

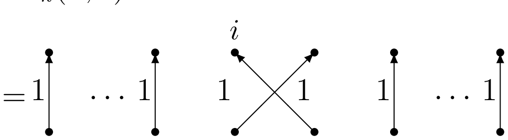
Proposition 4.1. *The dimension of $\mathcal{P}_k(x,r)$ is given by*

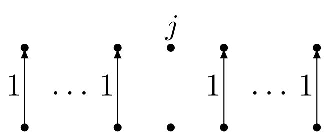
$$dim(\mathcal{P}_k(x,r)) = \sum_{i=1}^{2k} r^{2k-i} S(2k,i),$$

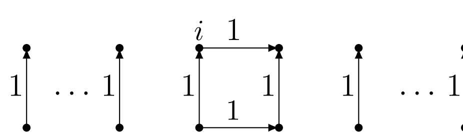
where $S(2k,i)$ is the Stirling number of the second kind, i.e. the number of set partitions of a set with $2k$ elements into i blocks.

Consider the following elements in $\mathcal{P}_k(x,r)$:

$t =$ 

$\tilde{s}_i =$ 

$\tilde{p}_j =$ 

$\tilde{p}_{i+\frac{1}{2}} =$ 

Note that $\tilde{s}_i$, $\tilde{p}_j$ and $\tilde{p}_{i+\frac{1}{2}}$ correspond to the generators of the Partition algebra in a natural way.

Proposition 4.2. *The elements $\{\tilde{s}_i, t, \tilde{p}_j, \tilde{p}_{i+\frac{1}{2}}\}$ generate $\mathcal{P}_k(x,r)$ and the elements $\tilde{s}_i, \tilde{p}_j, \tilde{p}_{i+\frac{1}{2}}$ satisfy the relations of Proposition 3.1 and in addition we have:*

$$t^r = 1, \qquad t\tilde{s}_1 t\tilde{s}_1 = \tilde{s}_1 t\tilde{s}_1 t, \qquad t\tilde{s}_i = \tilde{s}_i t \quad \textit{for } i > 1;$$

$$\tilde{p}_1 t = t\tilde{p}_1 = \tilde{p}_1, \qquad \tilde{p}_j t = t\tilde{p}_j, \quad \textit{for } j > 1.$$

4.1. **Schur-Weyl duality.** Using the permutation representation of $G_{n,r}$ on W, see equation (1), we can define an action of $G_{n,r}$ on $W^{\otimes k}$ for any non-negative integer k via the diagonal action:

$$g.(v_{(i_1,\epsilon^{j_1})}\otimes\cdots\otimes v_{(i_k,\epsilon^{j_k})}) = v_{g.(i_1,\epsilon^{j_1})}\otimes\cdots\otimes v_{g.(i_k,\epsilon^{j_k})}.$$

There is an action of $\mathcal{P}_k(n,r)$ on $W^{\otimes k}$ that commutes with the diagonal action of $G_{n,r}$. We define the action of a colored partition diagram as follows: let $d\in\mathcal{P}_k(n,r)$, we first label the top vertices $1,\ldots,k$ from left to right and the bottom vertices $k+1,\ldots,2k$ from left to right also. Define a function $\Phi:\mathcal{P}_k(n,r)\to \mathrm{End}(W^{\otimes k})$ as follows:

$$\begin{aligned}\Phi(d) &= (\Phi(d)^{(i_1,\epsilon^{j_k}),\ldots,(i_k,\epsilon^{j_k})}_{(i_k+1,\epsilon^{j_{k+1}}),\ldots,(i_{2k},\epsilon^{j_{2k}})})\\ &= ((d)^{i_1,\ldots,i_k}_{i_k+1,\ldots,i_{2k}})\delta_{j_{l_1}+a_1,j_{l_2}}\cdots\delta_{j_{l_m}+a_m,j_{l_m}}\end{aligned}$$

where $((d)^{i_1,\ldots,i_k}_{i_{k+1},\ldots,i_{2k}})$ is the matrix defined in Section 3, equation (6) for the partition algebra. We also have that in the colored diagram d, vertex l_1 and vertex l_2 are connected and the arrow points from l_1 to l_2, and the label of this edge is g_1, similarly we can do this for every pair of vertices that are connected, in our formula we assume that there are m edges. Therefore, we can define an action of $\mathcal{P}_k(n,r)$ on $W^{\otimes k}$ as follows:

$$d.(v_{(i_1,\epsilon^{j_1})}\otimes\cdots\otimes v_{(i_k,\epsilon^{j_k})}) = \Phi(d)(v_{(i_1,\epsilon^{j_1})}\otimes\cdots\otimes v_{(i_k,\epsilon^{j_k})}).$$

For the proof of the following Theorem see [2] Theorem 6.6.

Theorem 4.3. *$G_{n,r}$ and $\mathcal{P}_k(n,r)$ generate full centralizers of each other in End$(W^{\otimes k})$. That is, for $n\geq 2k$, we have*

(i) $\mathcal{P}_k(n,r)\cong End_{Gn,r}(W^{\otimes k})$.
(ii) $\mathbb{C}G_{n,r}$ *generates* $End_{\mathcal{P}_k(n,r)}(W^{\otimes k})$.

4.2. **Irreducible representations of $\mathcal{P}_k(n,r)$.** The colored partition algebra $P_{k-1}(n,r)$ is embedded in $\mathcal{P}_k(n,r)$ by adding a horizontal edge connecting the k-th and the k'-th vertex and labelling this edge by 1. We have the following tower of algebras

$$\mathcal{P}_0(n,r)\subset\mathcal{P}_1(n,r)\subset\mathcal{P}_2(n,r)\subset\cdots.$$

In this section we will describe the branching rules for this tower of partition algebras. We will describe an indexing set for the irreducible representations and describe the Bratteli diagram which encodes the induction and restriction rules.

Recall from Section 1.2 that the irreducible representations of $G_{n,r}$ are indexed by r-tuples of partitions of total size n. By Theorem 4.3 we have that $\mathcal{P}_k(n,r)$ is isomorphic to the centralizer algebra over the tensor product $W^{\otimes k}$, where W is the permutation module defined in Section 1.4. Hence, in order to decompose $\mathcal{P}_k(n,r)$ we need to decompose $W^{\otimes k}$. From Corollary 2.5 we have that the rule for decomposing this tensor product is done in a recursive way using the Kronecker product rule:

$$W \otimes V_{\vec{\gamma}} \cong \bigoplus_{\vec{\delta}} V_{\vec{\delta}},$$

where $\vec{\delta}$ is obtained by removing a box from $\vec{\gamma}$ and then adding a box to the resulting r-tuple. Let $\vec{\gamma}^*$ be the r-tuple of partitions so that the first part of the first component of $\gamma^{(1)}$ has been removed.

Proposition 4.4. *The irreducible representations of $\mathcal{P}_k(n, r)$ are indexed by the same set as the irreducible $G_{n,r}$-modules in $W^{\otimes k}$. In particular, this set for $n \geq 2k$ is given by*

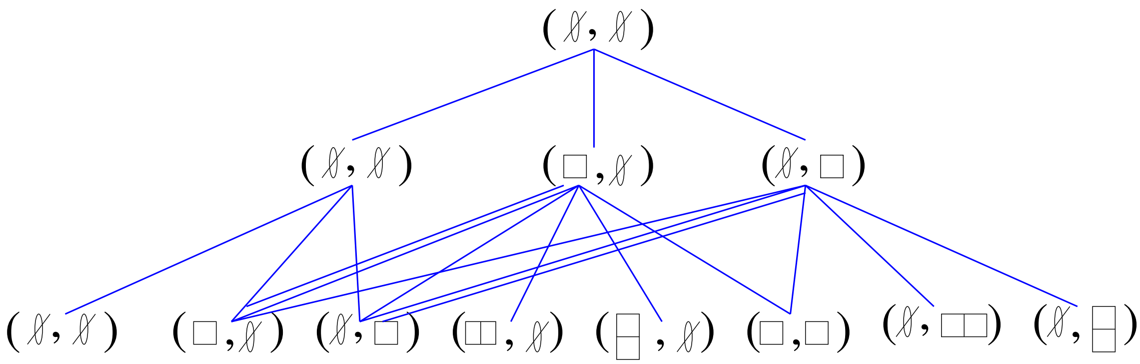
$$\Gamma(k, n, r) := \{\vec{\gamma} \vdash n \,:\, |\vec{\gamma}^*| \leq k\}$$

where $\vec{\gamma}^* = ((\gamma^{(1)})^*, \gamma^{(2)}, \ldots, \gamma^{(r)})$.

Proof. By the double centralizer theorem, all we have to show is that when we decompose the $G_{n,r}$-module $W^{\otimes k}$ in terms of irreducibles we get every partition in $\Gamma(k, n, r)$. This can be easily shown by induction on k. Notice that this is trivially true for $k = 0$, and for $k = 1$ it follows from Proposition 1.3. Now if we assume that $\Gamma(k-1, n, r)$ is an indexing set for $P_{k-1}(n, r)$, then we can see that according to the rule in Corollary 2.5 we can remove a box and then add it to the same exact position, hence every diagram in $\Gamma(k-1, n, r)$ is contained in $\Gamma(k, n, r)$, to obtain those diagrams such that $|\vec{\gamma^*}| = k$ we just remove a box from the the first row of $\gamma^{(1)}$ and add it in every possible way to the diagrams $\vec{\delta} \in \Gamma(k-1, n, r)$ such that $|\vec{\delta^*}| = k-1$. It is not possible to get a $\vec{\xi}$ such that $|\vec{\xi^*}| > k$ because we have added at most k boxes. $\square$

To construct the Bratteli diagram for the tower of colored partition algebras we label the vertices in the k-th level using the r-tuples in $\Gamma(k, n, r)$. There are edges from an r-tuple in level $k-1$, $\vec{\delta}$, to an r-tuple in level k, $\vec{\gamma}$, if it is possible to obtain $\vec{\gamma}$ from $\vec{\delta}$ by first removing one box from $\vec{\delta}$ and then adding one box to the resulting diagram. The number of edges from $\vec{\delta}$ to $\vec{\gamma}$ is the multiplicity of $V_{\vec{\gamma}}$ in $V_{\vec{\delta}} \otimes W$.

Below we give an example of the first three levels of the Bratteli diagram for $\mathcal{P}_k(n, r)$ where $r = 2$. In the diagram we have removed the first row from every $\vec{\gamma} \in \Gamma(k, n, r)$, that is we have indexed the vertices by $\vec{\gamma}^*$ instead of $\vec{\gamma}$. Notice that there are two edges from $((n-1), (1))$ to itself and also two edges from $((n-1, 1), \emptyset)$. In general, every r-tuple will occur such that $0 \leq |\vec{\gamma}| \leq k$ in level k by Proposition 4.4.



Bratteli diagram for $\mathcal{P}_0(n,2) \subset \mathcal{P}_1(n,2) \subset \mathcal{P}_2(n,2)$.

Recall that for the partition algebra every partition λ such that $0 \leq |\lambda| \leq k$ occurs in the decomposition of $\mathcal{P}_k(n)$, thus in this way $\mathcal{P}_k(n,r)$ can be seen as the natural generalization of $\mathcal{P}_k(n)$ to the complex reflection groups. To obtain the dimension for the irreducible representation, $N^{\vec{\gamma}}$ of $\mathcal{P}_k(n,r)$ we jut count the number of paths from $(\emptyset,\dots,\emptyset)$ to $\vec{\gamma}$ in the k-th level of the Bratteli diagram. We summarize these results on the following theorem that is a direct consequence of the Centralizer Theorem, Theorem 4.3 and Corollary 2.5.

Theorem 4.5. *Let $n,k,r \in \mathbb{Z}_{\geq 0}$. Let $V_{\vec{\gamma}}$ denote the irreducible $G_{n,r}$-module indexed by $\vec{\gamma}$ and let $N^{\vec{\delta}}$ index the irreducible representation of $\mathcal{P}_k(n,r)$ indexed by $\vec{\delta}$. Then*

(a) *The dimension of the irreducible representation $N^{\vec{\gamma}}$ of $\mathcal{P}_k(n,r)$ is equal to the multiplicity of $V_{\vec{\gamma}}$ in $W^{\otimes k}$. This number is equal to the number of paths from the top of the Bratteli diagram to $\vec{\gamma}$.*

(b) *The decomposition of $W^{\otimes k}$ as a bimodule for $G_{n,r} \times \mathcal{P}_k(n,r)$ is*

$$W^{\otimes k} \cong \bigoplus_{\vec{\gamma}\in\Gamma(n,k,r)} V_{\vec{\gamma}} \otimes N^{\vec{\gamma}}.$$

5. A subalgebra of the Partition Algebra

In [26] Tanabe introduced a subalgebra of the partition algebra by considering the following situation. Let U be the monomial representation of the complex reflection group $G_{n,r}$. Then define the centralizer algebra

$$\mathrm{End}_{G_{n,r}}(U^{\otimes k}).$$

The matrices corresponding to the representation U are the $n \times n$ permutation matrices such that the non-zero entry in each row and column is an r-th root of unity. The dimension of this representation is n and it is known to be isomorphic to $V_{((n-1),(1),\emptyset,\dots,\emptyset)}$.

If B is a block of a set partition d define

$$\kappa(B) = |\{(\#\text{ top vertices in } B) - (\#\text{ bottom vertices of } B)\}|,$$

and let

$$A_{k,r} = \{d \in A_k \,:\, \text{for all blocks } B \text{ of } d,\ \kappa(B) \equiv 0 \bmod r\}.$$

Then the algebra of Tanabe can be defined by

$$\mathcal{T}_k(n,r) := \mathbb{C}\text{-span}\{d \mid d \in A_{k,r}\}.$$

$\mathcal{T}_k(n,r)$ is a subalgebra of $\mathcal{P}_k(n)$. In fact,

$$\mathcal{T}_k(n,1) = \mathcal{P}_k(n).$$

Furthermore, if we let r go to infinity, we have

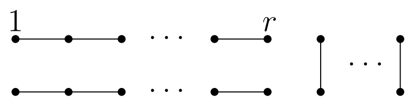
$$\mathcal{U}_k := \mathcal{T}_k(n,\infty) = \{d \in A_k \,:\, \kappa(B) = 0 \text{ for all blocks } B \text{ of } d\}.$$

The algebras $\mathcal{U}_k$ are the algebras studied by Kosuda [11, 12] and these algebras do not depend on the parameter n.

Define the following elements in $\mathcal{P}_k(n)$:

$$f_r := p_{\frac{1}{2}} p_{\frac{3}{2}} \cdots p_{\frac{r-1}{2}} p_1 p_2 \cdots p_r p_{\frac{1}{2}} p_{\frac{3}{2}} \cdots p_{\frac{r-1}{2}}.$$

Here, it is understood that if $r > k$, then $f_r := 0$. This element can be represented as follows when $r \le k$ using partition diagrams.

$f_r =$ 

Proposition 5.1 ([26],Lemma 3.1)**.** *$\mathcal{T}_k(n,r)$ is generated by $s_1, \ldots, s_{k-1}$, $p_{\frac{3}{2}}$ and f_r.*

The generator f_r satisfy the following relations:

$$f_r^2 = nf_r, \qquad p_{\frac{3}{2}} = f_r p_{\frac{3}{2}}, \qquad s_i f_r = f_r s_i, \quad \text{for } i \neq r.$$

In addition if we let $f_{i,r}$ denote the partition diagram that corresponds to the set partition $\{\{i,\ldots,i+r-1\}\{i',\ldots,i'+r-1\}\}$, then we have $f_r = f_{1,r}$ and

$$s_1 s_2 \cdots s_r f_{i,r} s_r \cdots s_2 s_1 = f_{i+1,r}.$$

Remark: The Temperley-Lieb algebra and the Brauer algebra are subalgebras only for $\mathcal{T}_k(n,2)$, when we set $r = 2$. In the case $r = 2$, the $f_{i,2}$ are the well-known generators of the Temperley-Lieb algebra.

Let $\tilde{S}(2k,l)$ be the number of set partitions in $A_{k,r}$ into l blocks. Notice that in general, $\tilde{S}(2k,l)$ is less than or equal to the Stirling number of the second kind. Hence,

$$\dim(\mathcal{T}_k(n,r)) = \sum_{l=1}^{2k} \widetilde{S}(2k,l).$$

Remark: In the case that $r = 2$ we have that the dimension of $\mathrm{End}_{G_{n,2}}(U^{\otimes k})$ is the number of set partitions such that all the blocks have even cardinality, in this case the formula has the following simple recursive description. Let d_k denote the dimension of $\mathrm{End}_{G_{n,2}}(U^{\otimes k})$, then

$$d_k = \sum_{i=1}^{k} \binom{2k-1}{2i-1} \cdot d_{k-i}.$$

Futhermore, the exponential generating function for this sequence is $exp(cos(x)-1)$. This is sequence A005046 in the *Encyclopedia of Integer Sequences* [24].

There is a recursion formula for the dimensions of $\mathcal{U}_k$. Let u_k denote the dimension of $\mathcal{U}_k$. The formula can easily be derived by counting set partitions so that each block contains the same number of "top" vertices as "bottom" vertices.

$$u_k := \sum_{1^{m_1}\dots n^{m_k} \vdash k} \left(\frac{k!}{(1!)^{m_1}\cdots(n!)^{m_k}} \right)^2 \frac{1}{m_1!\cdots m_k!} \tag{7}$$

where the sum runs over all partitions of k. Starting at $k=0$, the first values are

$$1, 1, 3, 16, 131, 1496, 22482, \dots$$

This is sequence A023998 in [24]. These numbers and generalizations are studied in [25]; in particular, the following recursion is given in [25, equation (11)]:

$$u_{k+1} = \sum_{i=0}^{k} \binom{k}{i} \binom{k+1}{i} u_i\,, \quad u_0 = 1\,.$$

Although there is no known explicit formula for $\dim(\mathcal{T}_k(n,r))$, we do know that

$$\lim_{r\to\infty} \dim(\mathcal{T}_k(n,r)) = u_k.$$

5.1. **Schur-Weyl Duality for $\mathcal{T}_k(n,r)$.** The algebra $\mathcal{T}_k(n,r)$ was constructed by Tanabe as a centralizer algebra of the complex reflection group $G_{n,r}$. Let $U = \mathbb{C}^n$, and let $v_1, \dots, v_n$ be its standard basis vectors. There is a natural action of $G_{n,r}$ on U:

$$t.v_i = \begin{cases} \epsilon v_1 & \text{if } i = 1 \\ v_i & \text{otherwise,} \end{cases}$$

where $\epsilon = e^{2\pi i/r}$. And

$$\sigma.v_i = v_{\sigma(i)} \qquad \text{for } \sigma \in S_n.$$

This representation is called the *monomial representation* of $G_{n,r}$. Using this representation we can define a diagonal action of $G_{n,r}$ on $U^{\otimes k}$ via the diagonal action.

$$g(v_{i_1} \otimes \cdots \otimes v_{i_,}) = g\cdot v_{i_1} \otimes \cdots \otimes g \cdot v_{i_k}, \qquad \text{for } g \in G_{n,r}.$$

Remark: It is well-known that the monomial representation of $G_{n,r}$ is irreducible and furthermore, it is indexed by the r-tuple of partitions $((n-1),(1),\emptyset,\dots,\emptyset)$.

For any element $X \in \mathrm{End}_{G_{n,r}}(U^{\otimes k})$ we denote by $(X^{b_1,\cdots,b_k}_{a_1,\cdots a_k})$, $1 \le a_i, b_i \le n$, its matrix of coefficients with respect to the basis $\{v_{i_1} \otimes \cdots \otimes v_{i_k} \,:\, 1 \le i_j \le n\}$ of $U^{\otimes k}$.

The following lemma is Lemma 3.2 in [26].

Lemma 5.2. *$X \in End(U^{\otimes k})$ if and only if $X^{b_1,\dots,b_k}_{a_1,\dots,a_k} = X^{\sigma(b_1)),\dots,\sigma(b_k)}_{\sigma(a_1),\dots,\sigma(a_k)}$ for any permutation σ and $|\{i \ : a_i = j\}| \equiv |\{i \,:\, b_i = j\}|$ mod r for all $j = 1, \dots, n$ and $i = 1, \dots, k$.*

Proof. For $X \in \mathrm{End}_{G_{n,r}}(U^{\otimes k})$ and $\sigma \in S_n$ we have

$$\sigma^{-1} X \sigma (v_{a_1} \otimes \cdots \otimes v_{a_k}) = \sum_{b_1,\dots,b_k \in [n]} X^{\sigma(b_1),\dots,\sigma(b_k)}_{\sigma(a_1),\dots,\sigma(a_k)} v_{b_1} \otimes \cdots v_{b_k}.$$

We also have for the generator t:

$$t^{-1} X t (v_{a_1} \otimes \cdots \otimes v_{a_k}) = \sum_{b_1,\dots,b_k \in [n]} \epsilon^{|\{i\,:\,a_i = j\}| - |\{i\,:\,b_i=j\}|} X^{b_1,\dots,b_k}_{a_1,\dots a_k} v_{b_1} \otimes \cdots \otimes v_{b_k}.$$

Since $\epsilon^{|\{i\,:\,a_i=j\}|-|\{i\,:\,b_i=j\}|} = 1$ if and only if $|\{i \,:\, a_i = j\}| - |\{i \,:\, b_i = j\}| \equiv 0 \bmod r$ we have the second condtion of the lemma. The result follows in general since $G_{n,r}$ is generated by t and the permutations in S_n. $\square$

Lemma 5.2 implies that X commutes with the action of $G_{n,r}$ on $U^{\otimes k}$ if and only if the matrix entries are equal on $G_{n,r}$-orbits on the $2k$ cartesian product $\{1, \dots, n\}^{\times 2k}$. These orbits are in one-to-one correspondence with set partitions $B = \{B_1, \dots, B_s\}$ of $\{1, \dots, 2k\}$ such that $|B_i \cap \{1, \dots, k\}| \equiv |B_i \cap \{k+1, \dots, 2k\}| \bmod r$, for all $i = 1, \dots, s$.

Example: For $r = 2$, we have that $G_{n,2}$ is the hyperoctahedral group, in this case the $G_{n,2}$-orbits are in one-to-one correspondence with the set of partitions such that all blocks have even cardinality.

Notice that every set partition will have at most n blocks and each set partition gives rise to an equivalence relation, i.e. two elements $i, j \in \{1, 2, \dots, 2k\}$ are equivalent if and only if they belong to the same block. This is equivalent to requiring that $a_i = a_j$ where we have identified $b_l = a_{l+k}$ for simplicity.

For each set partition B we define the matrix M_B

$$(M_B)^{a_1,\dots,a_k}_{a_{k+1},\dots,a_{2k}} = \begin{cases} 1 & \text{if } a_i = a_j \text{ if and only if } i \text{ and } j \text{ are in the same block.} \\ 0 & \text{otherwise.} \end{cases}$$

Notice that M_B will be zero if the number of blocks in B is greater than n, thus as long as $2k \le n$ we will always get all these matrices to be nonzero.

Another way to write the M_B is in terms of the elementary matrices $E^{i_1,\dots,i_k}_{i_{k+1},\dots,i_{2k}}$, these are the $n^k \times n^k$ matrices with a 1 in the $(i_1, \dots, i_k), (i_{k+1}, \dots, i_{2k})$ position and zeros everywhere else. Then we have

$$M_B = \sum E^{i_1,\dots,i_k}_{i_{k+1},\dots,i_{2k}},$$

where the sum is over all $1 \leq i_1, \dots, i_{2k} \leq n$ with the condition that $i_a = i_b$ if and only if a, b are in the same block of B. Hence we have the following proposition.

Proposition 5.3. *The set* $\{M_B \neq 0 \,:\, B \in A_{k,r}\}$ *form a basis for* $End_{G_{n,r}}(U^{\otimes k})$.

The following is a reformulation of the results obtained in [26].

Theorem 5.4. $G_{n,r}$ *and* $\mathcal{T}_k(n,r)$ *generate full centralizers of each other in* $End(U^{\otimes k})$. *In other words, for* $n \geq 2k$*, we have the following*

(i) $\mathcal{T}_k(n,r) \cong End_{G_{n,r}}(U^{\otimes k})$,

(ii) $G_{n,r}$ *generates* $End_{\mathcal{T}_k(n,r)}(U^{\otimes k})$.

A special case of 5.4 is when r is very large, in this case we obtain

Corollary 5.5. *There is a right action of* $\mathcal{U}_k$ *on* $U^{\otimes n}$ *determined by*

$$(v_{i_1} \otimes \cdots \otimes v_{i_n}) \cdot b_j = \delta(i_j, i_{j+1}) v_{i_1} \otimes \cdots \otimes v_{i_n} \quad \text{and} \quad (v_{i_1} \otimes \cdots \otimes v_{i_n}) \cdot \sigma = v_{i_{\sigma(1)}} \otimes \cdots \otimes v_{i_{\sigma(n)}}$$

for $1 \leq i \leq n-1$ *and* $\sigma \in S_n$. *This action commutes with the left action of* $G_{n,r}$ *on* $U^{\otimes n}$. *Moreover, if* $m \geq 2n$ *and* $r > n$ *then the resulting map*

$$\mathcal{U}_k \to \mathrm{End}_{G_{n,r}}(U^{\otimes k}) \tag{8}$$

is an isomorphism of algebras.

5.2. **The Irreducible representations of $\mathcal{T}_k(n,r)$.** The monomial representation of $G_{n,r}$ denoted by U in Section 5.1 is known to be isomorphic to the irreducible representation $V_{((n-1),(1),\emptyset,\dots,\emptyset)}$. Then by Theorem 2.3 we have that

$$U \otimes V_{\vec{\gamma}} \cong \bigoplus_{\vec{\delta}} V_{\vec{\delta}},$$

where the sum is over all r-tuples of Young diagrams that are obtained from removing a box from $\gamma^{(i)}$ and then adding it to $\gamma^{(i+1)}$, for any possible $i = 1, \dots, r-1$. And if we remove a box from $\gamma^{(r)}$, then we add it to $\gamma^{(1)}$.

Proposition 5.6. *The irreducible representations of* $\mathcal{T}_k(n,r)$ *are indexed by the same set as the irreducible* $G_{n,r}$*-modules in* $U^{\otimes k}$. *This set can be recursively described as follows: Let* $\Omega_0(n,r) := \{((n), \emptyset, \dots, \emptyset)\}$ *be the indexing set for* $\mathcal{T}_0(n,r)$ *and for* $k \geq 1$ *let* $\Omega_k(n,r)$ *denote the index set for the irreducible representations of* $\mathcal{T}_k(n,r)$. *Then the set* $\Omega_{k+1}(n,r)$ *can be constructed from* $\Omega_k(n,r)$ *as follows:* $\vec{\gamma} \in \Omega_{k+1}(n,r)$ *if there exists a* $\vec{\delta} \in \Omega_k(n,r)$ *such that* $\vec{\gamma}$ *is the result of removing a box from* $\delta^{(i)}$ *and then adding a box to* $\delta^{(i+1)}$ *for* $i = 1, \dots r-1$*, or removing a box from* $\delta^{(r)}$ *and then adding a box to* $\delta^{(1)}$.

This proposition is a direct consequence of the double centralizer theorem, Theorem 5.4 and Theorem 2.3 for $k = 2$.

Example. Let $r = 2$, then $\Omega_k(n,r)$ has an easy closed form

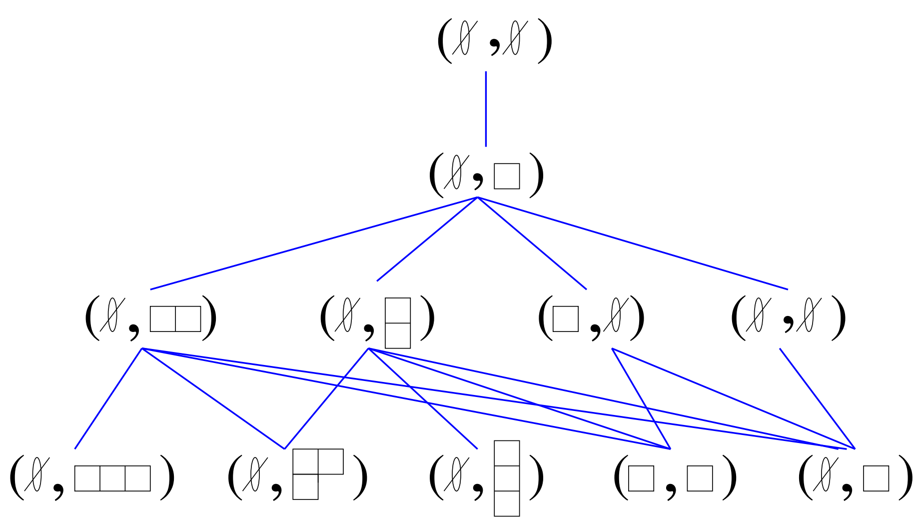
$$\Omega_k(n,2) := \{((n-j,\alpha),\beta) \,:\, \beta \vdash k-2i, \alpha \vdash m, 0 \leq m \leq i, 0 \leq i \leq \lfloor \frac{j}{2} \rfloor, j = k+m-2i\}.$$

Remark: If $n \geq k$, then it is always possible to remove a box from the first coordinate and add it to the second coordinate, hence every diagram of the form $(\emptyset, \lambda, \emptyset, \ldots, \emptyset)$ for $\lambda \vdash k$ is an element in $\Omega_k(n,r)$. These representations are isomorphic to the irreducible representations of the symmetric group.

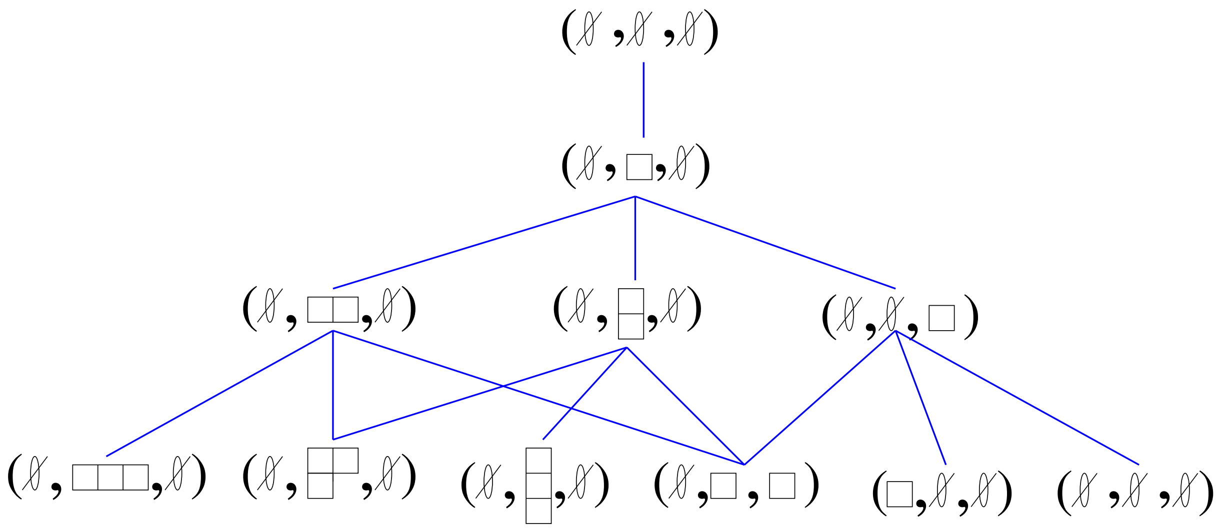
Using this rule we can now recursively construct the Bratteli diagram for the tower of algebras

$$\mathcal{T}_0(n,r) \subset \mathcal{T}_1(n,r) \subset \mathcal{T}_2(n,r) \subset \mathcal{T}_3(n,r) \subset \cdots .$$

The vertices in the k-th level are indexed by $\Omega_k(n,r)$ and there is an edge from $\vec{\delta}$ in the k-th level to $\vec{\gamma}$ in the $k+1$-st level if it is possible to obtain $\vec{\gamma}$ from $\vec{\delta}$ using rule of removing a box from the i-th coordinate and adding a box to the $i+1$-st, with the understanding that if $i = r$, then we add the box to the first coordinate. In the examples below we give the first few levels of the Bratteli diagrams for $\mathcal{T}_k(n,r)$ when $r = 2$ and $r = 3$. Notice that we have labelled the vertices of the Bratteli diagram with $\vec{\gamma}^*$ instead of $\vec{\gamma} \in \Omega_k(n,r)$.



Bratteli diagram for $\mathcal{T}_0(n,2) \subset \mathcal{T}_1(n,2) \subset \mathcal{T}_2(n,2) \subset \mathcal{T}_3(n,2)$.



Bratteli diagram for $\mathcal{T}_0(n,3) \subset \mathcal{T}_1(n,3) \subset \mathcal{T}_2(n,3) \subset \mathcal{T}_3(n,3)$.

In complete analogy with the Partition algebra we obtain the following theorem as a consequence of the Centralizer Theorem, Theorem 5.4 and Theorem 2.3.

Theorem 5.7. *Let $n, k, r \in \mathbb{Z}$. Let $V_{\vec{\gamma}}$ denote the irreducible $G_{n,r}$-module indexed by $\vec{\gamma}$ and let $M^{\vec{\delta}}$ index the irreducible representation of $\mathcal{T}_k(n,r)$ indexed by $\vec{\delta}$. Then*

(a) *The dimension of the irreducible representation $M^{\vec{\gamma}}$ of $\mathcal{T}_k(n,r)$ is equal to the multiplicity of $V_{\vec{\gamma}}$ in $U^{\otimes k}$. This number is equal to the number of paths from the top of the Bratteli diagram to $\vec{\gamma}$.*

(b) *The decomposition of $U^{\otimes k}$ as a bimodule for $G_{n,r} \times \mathcal{T}_k(n,r)$ is*

$$U^{\otimes k} \cong \bigoplus_{\vec{\gamma} \in \Omega(n,k,r)} V_{\vec{\gamma}} \otimes M^{\vec{\gamma}}.$$

Remark: Notice that the colored partition algebra $\mathcal{P}_k(n,r)$ contains a copy of the group algebra of $G_{k,r}$, while $\mathcal{T}_k(n,r)$ does not, it only contains an isomorphic copy of the symmetric group algebra $\mathbb{C}S_k$.

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