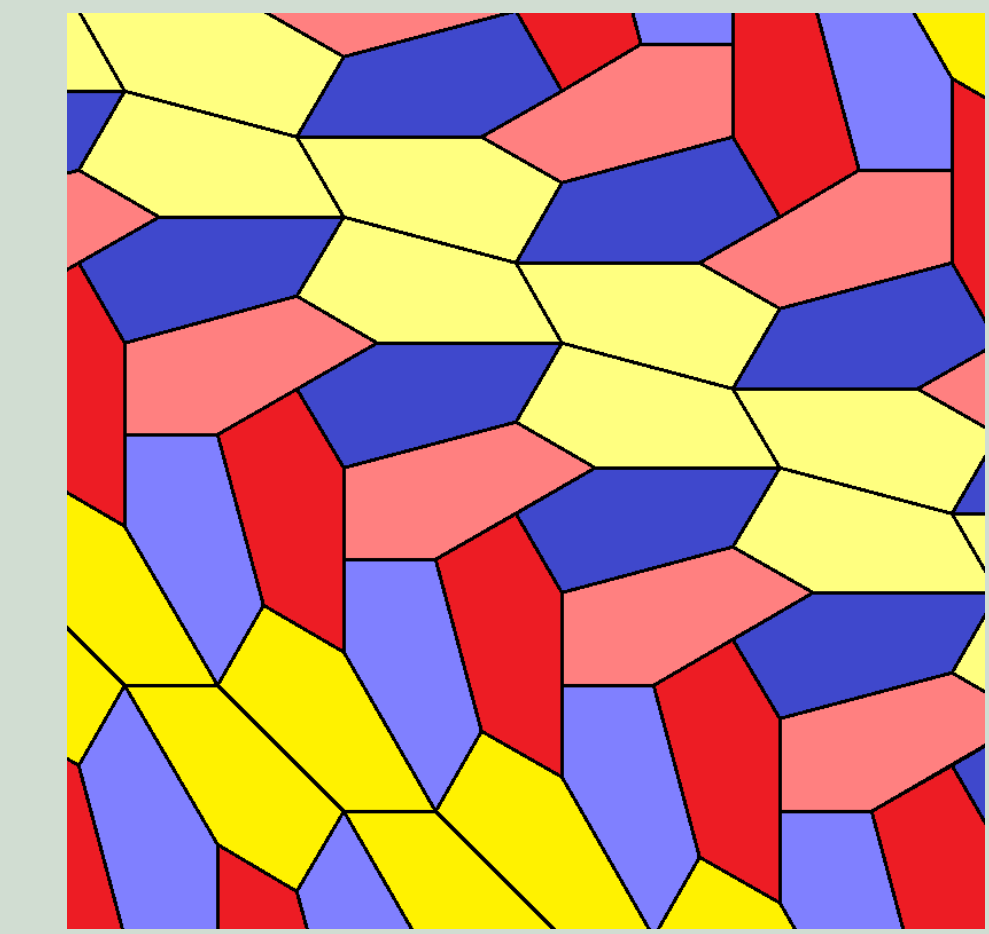# Pentagonal Tiling

Department of Mathematics, Dartmouth College

INTRODUCTION

A pentagonal tiling is a tiling of the plane where each individual piece is in the shape of a pentagon. Fifteen types of convex pentagons are known to tile the plane with one type of tile. This list has been shown to be complete by Michaël Rao.

The first five were discovered by German mathematician Karl Reinhardt in 1918. After a gap of 50 years, R. B. Kershner found three more in 1968. Richard James subsequently discovered a ninth type of pentagonal tiling in 1975 and over the next few years, Marjorie Rice discovered another four types. Rolf Stein found a 14th tiling in 1985. The most recently discovered 15th tiling was found by Casey Mann, Jennifer McLoud and David Von Derau of the University of Washington Bothell in 2015 using a computer to exhaustively search through a large but finite set of possibilities.
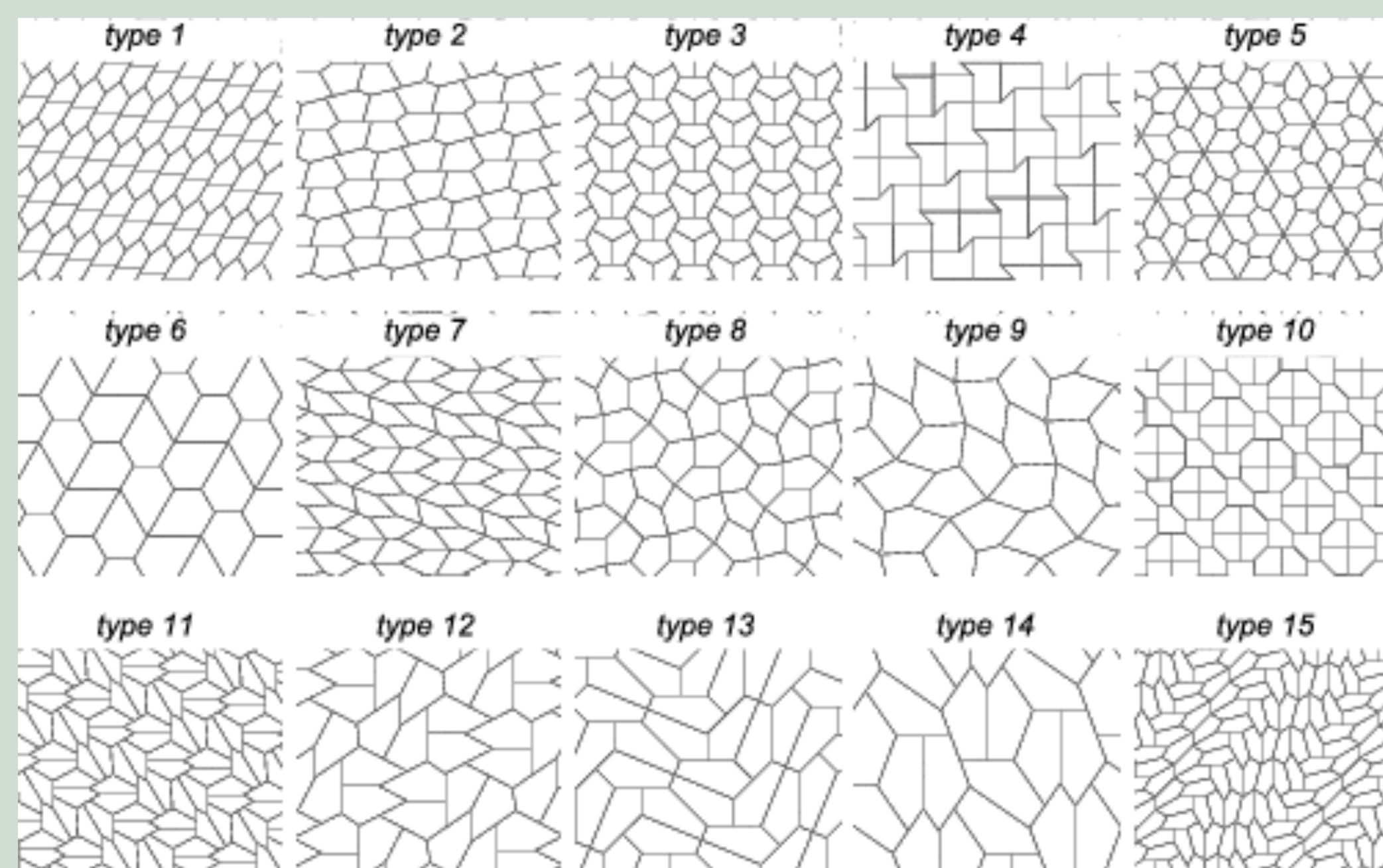

Figure 1. 15 Classes of Convex Pentagonal Tilings

SECRET BEHIND THE TILES

Among the 15 types of convex pentagons that can tile the plane monohedrally, each enumerated tiling family contains pentagons that belong to no other type; however, some individual pentagons may belong to multiple types. In addition, some of the pentagons in the known tiling types also permit alternative tiling patterns beyond the standard tiling exhibited by all members of its type.

The secret of these pentagons lies in the relationship between edges and angles. In the Figures below, the sides of length a, b, c, d, e are directly clockwise from angles at vertices A, B, C, D, E respectively.

Figure 2 (a - c). 15 Monohedral Pentagonal Tiles

1. $B+C=180°$, $A+D+E=360°$

2. $c=e$, $B+D=180°$

3. $a = b, d = c + e$, $A = C = D = 120°$

4. $b = c, d = e$, $B = D = 90°$

5. $a = b, d = e$, $A = 60°, D = 120°$

6. $a = d = e, b = c$, $B + D = 180°, 2B = E$

7. $b = c = d = e$, $B + 2E = 2C + D = 360°$

8. $b = c = d = e$, $2B + C + D = 360°$

9. $b = c = d = e$, $2A + C = D + 2E = 360°$

10. $a = b = c + e$, $A = 90°, B + E = 180°, B + 2C = 360°$

11. $2a + c = d = e$, $A = 90°, 2B + C = 360°$, $C + E = 180°$

12. $2a = d = c + e$, $A = 90°, 2B + C = 360°$, $C + E = 180°$

13. $d = 2a = 2e$, $B = E = 90°, 2A + D = 360°$

14. $2a = 2c = d = e$, $A = 90°, B ≈ 145.34°, C ≈ 69.32°, D ≈ 124.66°, E ≈ 110.68°$ $(2B + C = 360°, C + E = 180°)$.

15. $a = c = e, b = 2a$, $A = 150°; B = 60°; C = 135°; D = 105°, E = 90°$
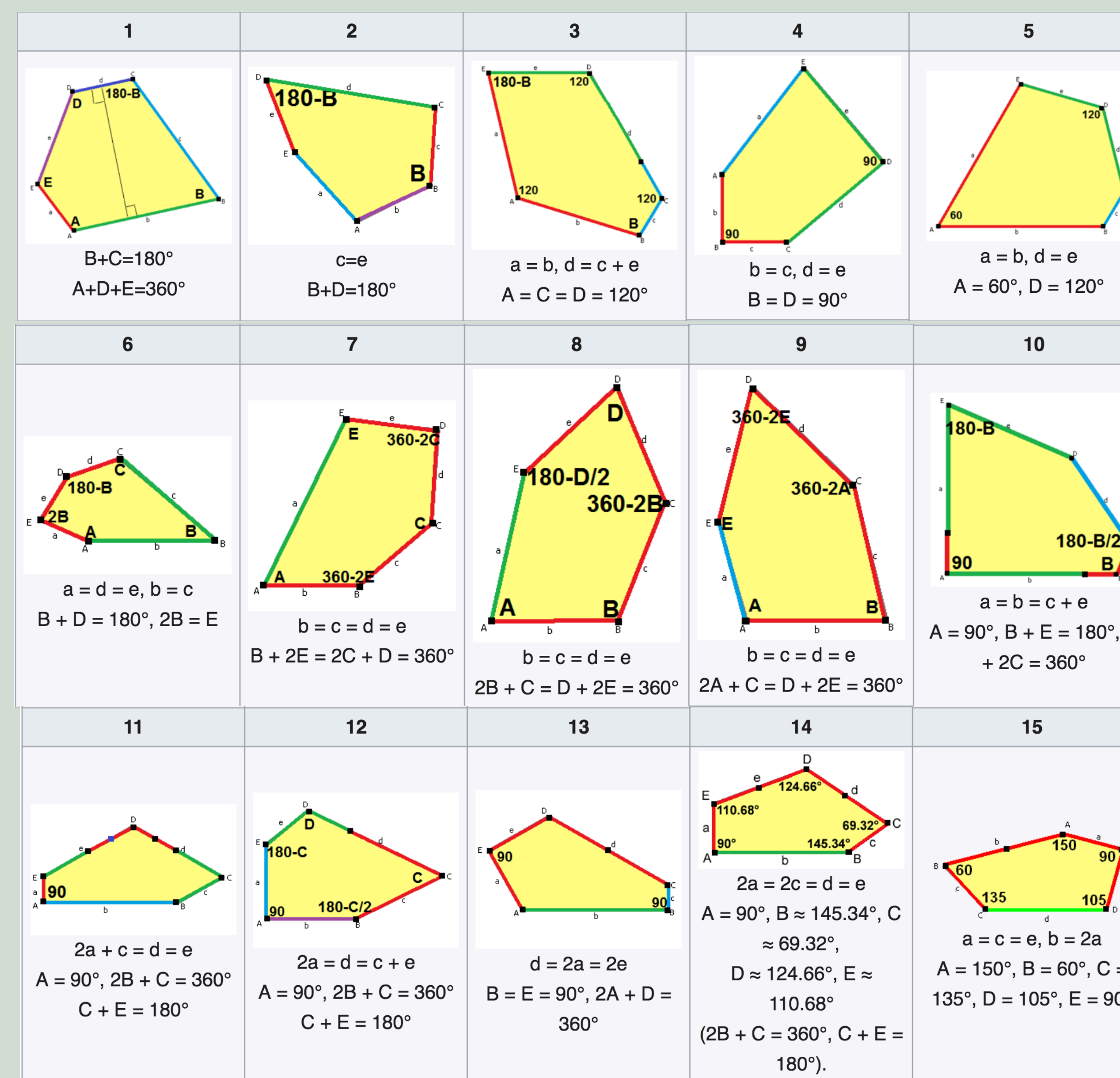
Figure 2 (a - c). 15 Monohedral Pentagonal Tiles

Many of these monohedral tile types have degrees of freedom. These freedoms include variations of internal angles and edge lengths. Types 1, 2, 4, 5, 6, 7, 8, 9, and 13 allow parametric possibilities with nonconvex prototiles.

Periodic tilings are characterised by their wallpaper group symmetry, for example p2(2222) is defined by four 2-fold gyration points.
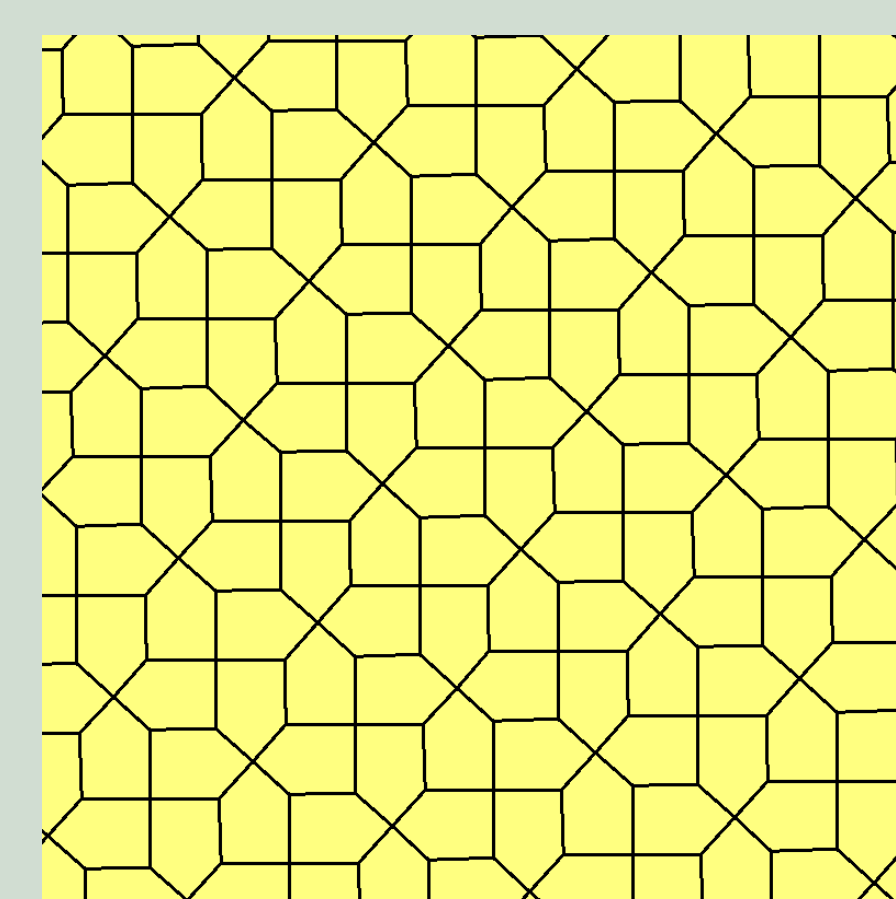
SPECIFIC TILING: TYPE 4

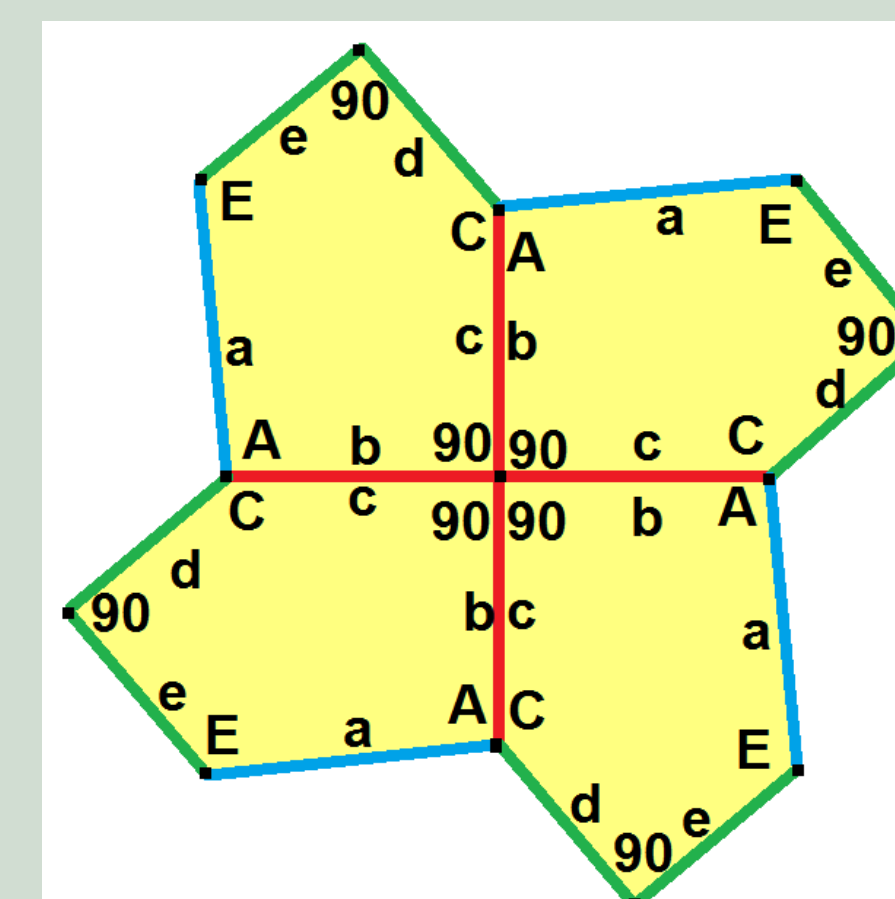

Figure 3 (a). Type 4, p4(442)

Figure 3 (b). 4-tile Primitive Unit of Type 4, p4(442)

What is a "Primitive Unit"?

Can you find larger units of connected pentagons that are repeated in the plane tiling? We call the smallest group of connected shapes that can fill up the plane by translation the "primitive unit." The numbers of pentagons that make up the primitive unit of different tilings might differ.
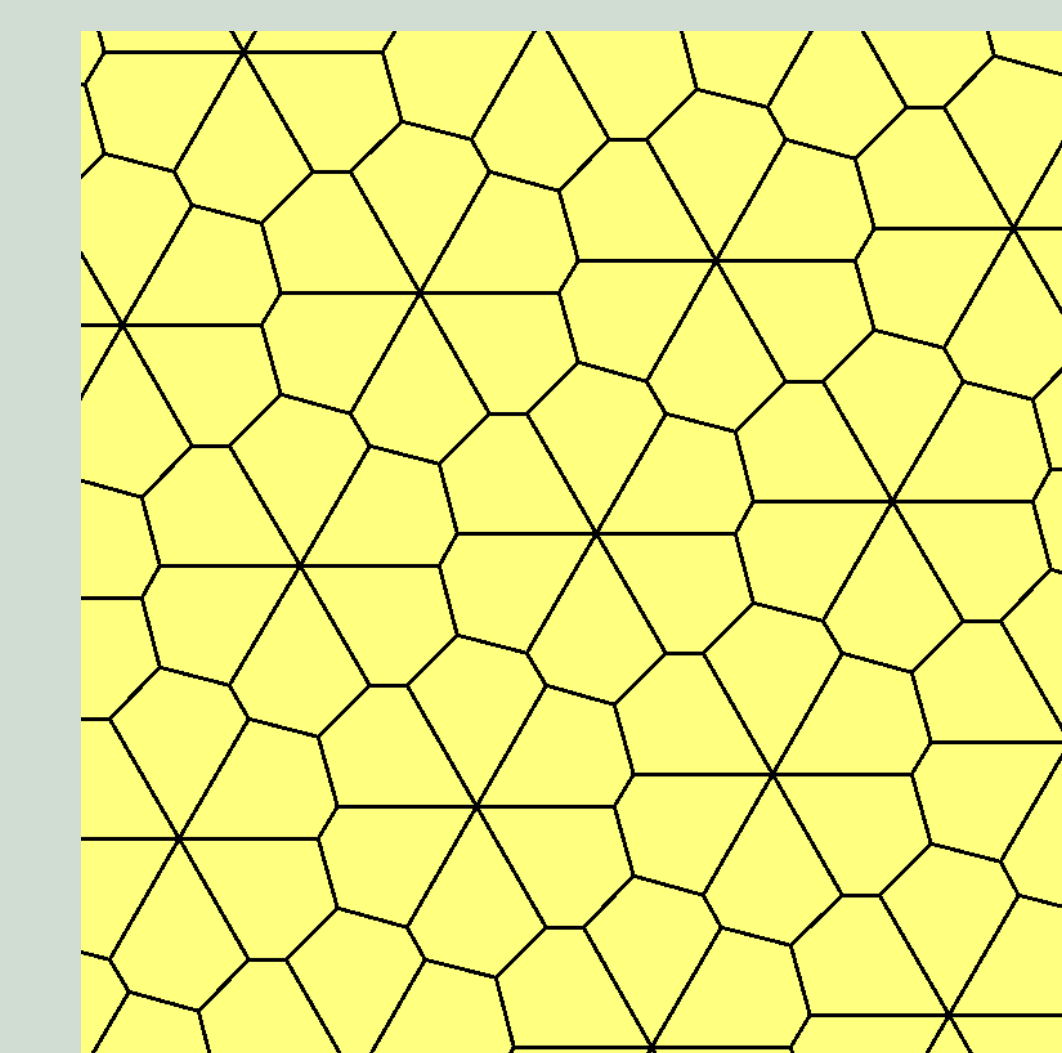
SPECIFIC TILING: TYPE 5

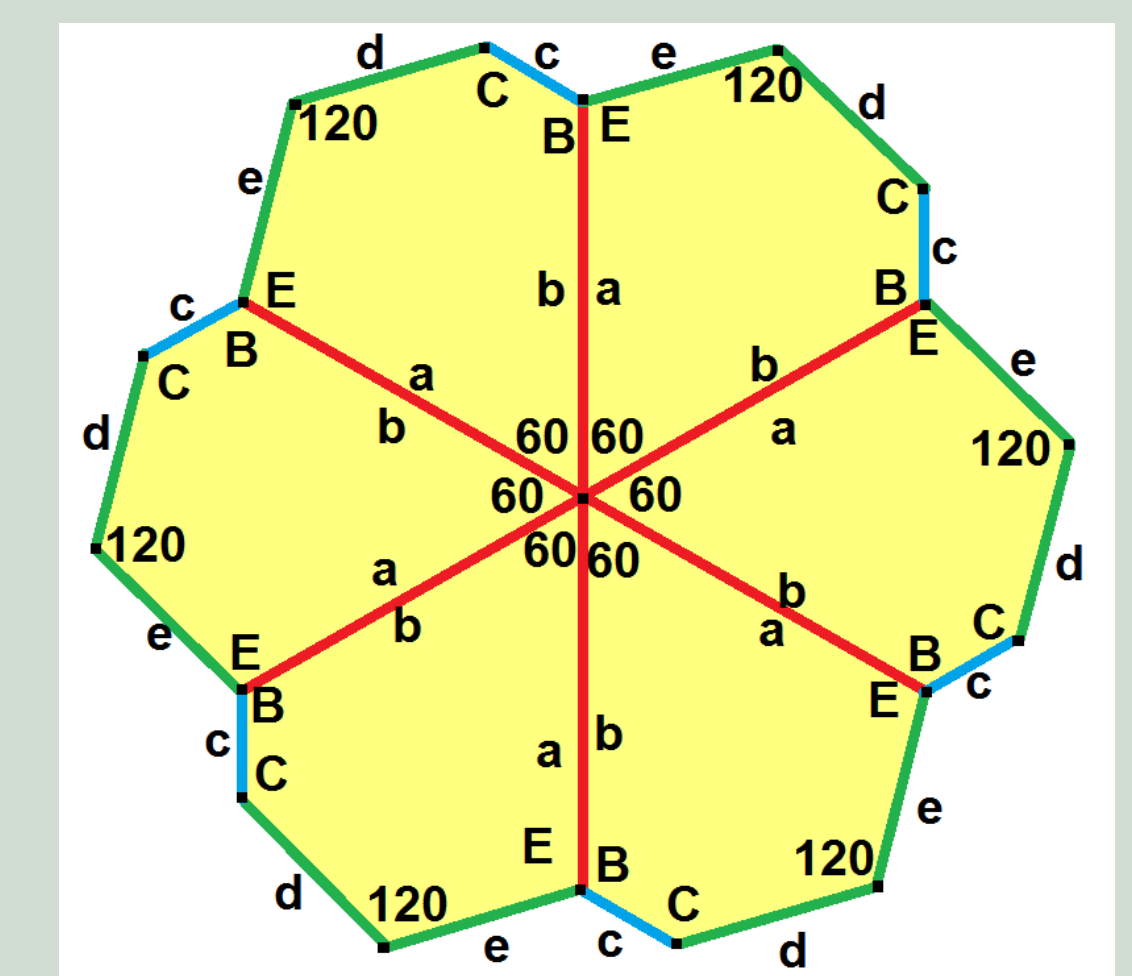
Figure 4 (a). Type 5, p6(632)

Figure 4 (b). 6-tile Primitive Unit of Type 5, p6(632)

SPECIFIC TILING: TYPE 15

Type 15 is perhaps the most complicated one. It has completely determined tiles, with no degrees of freedom. The primitive units contain twelve tiles respectively. It has pgg (22×) symmetry, and p2 (2222) if chiral pairs are considered distinct.
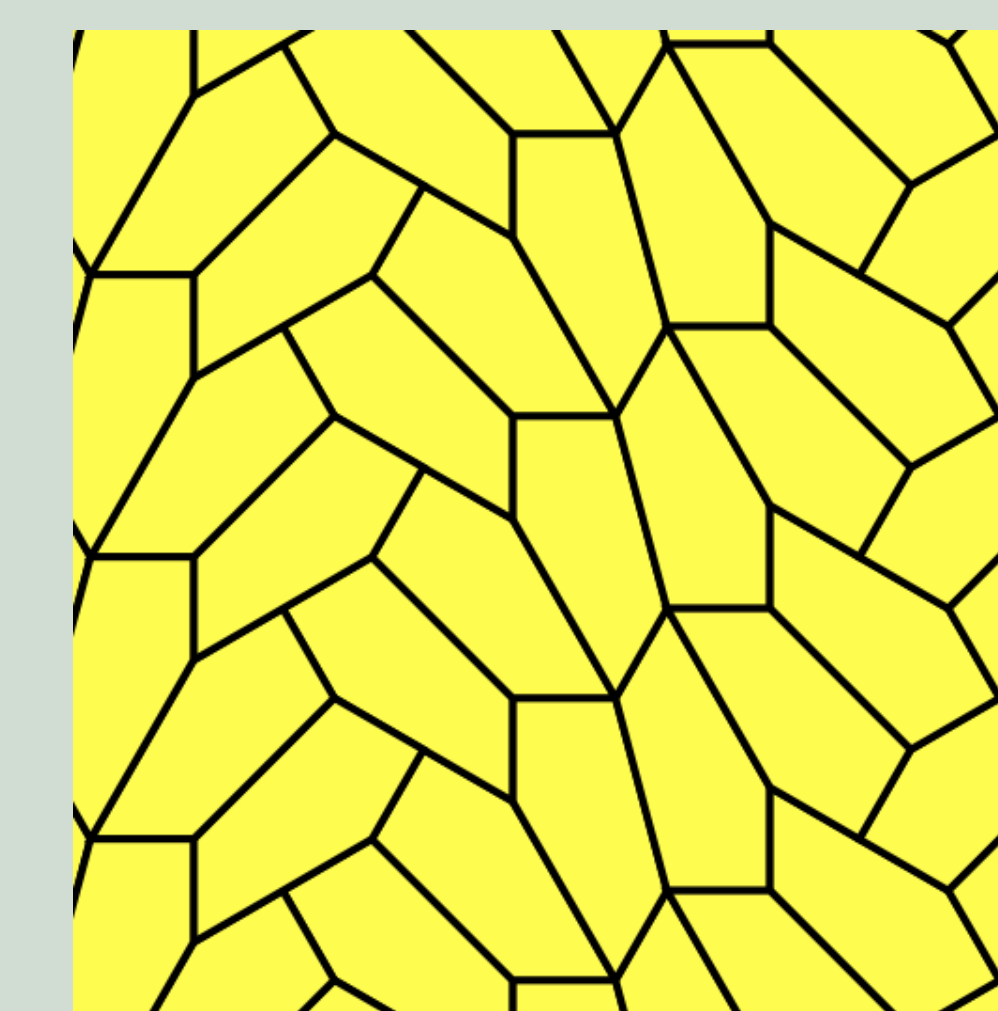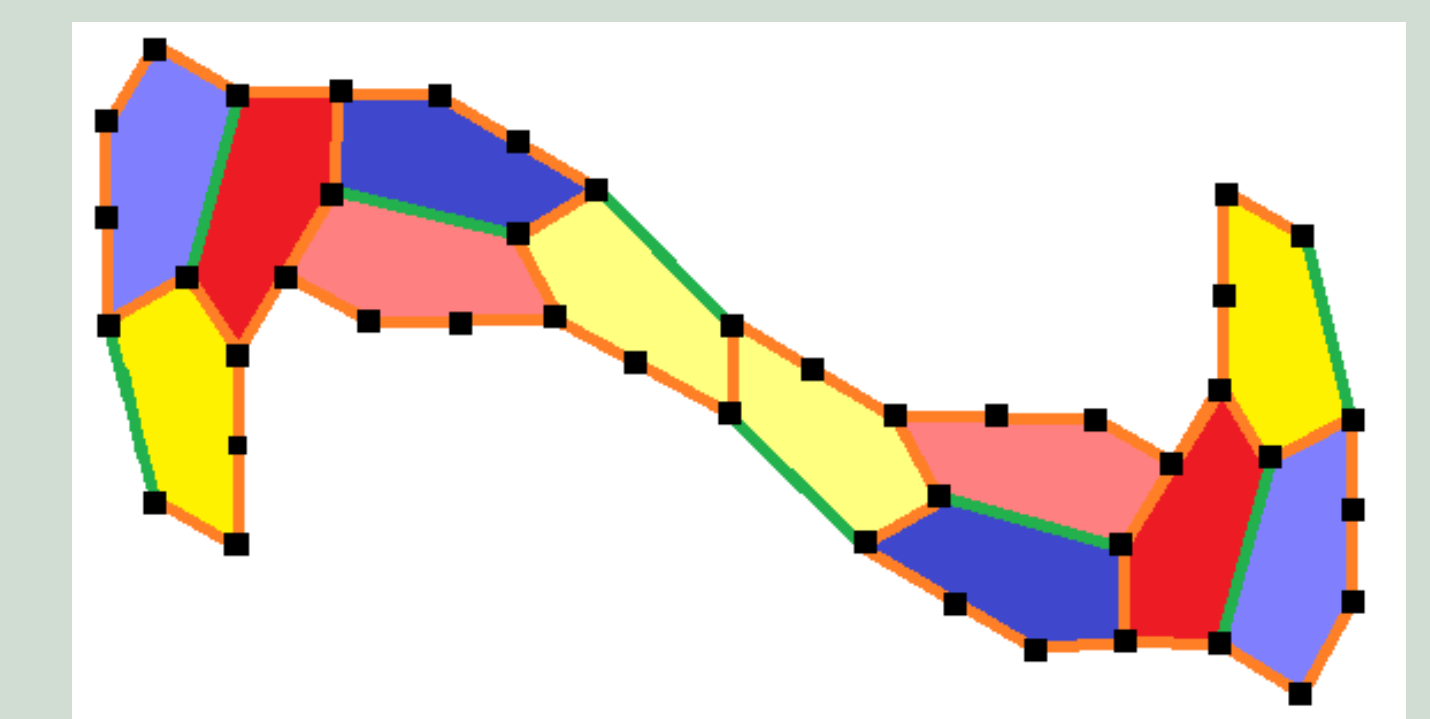

Figure 5 (a). Type 15

Figure 5 (b). 12-tile Primitive Unit of Type 15

REFERENCES

1. Weisstein, Eric W. "Pentagon Tiling." http://mathworld.wolfram.com/PentagonTiling.html
2. https://en.wikipedia.org/wiki/Pentagonal_tiling