Minimum and Maximum Points

November 28, 2007

Local Maximum and Local Minimum

Definition: A function f of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is in a neighborhood of (a, b). [This means that $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**.

If $f(x, y) \geq f(a, b)$ when (x, y) is in a neighborhood of (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

Critical Points

A point (a, b) is called a **critical point** (or stationary point) of f if

$$\nabla f(a, b) = \langle 0, 0 \rangle$$

or if of of this partial derivatives does not exist.

Local max and min points are critical points

Theorem: If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$, that is $\nabla f(a, b) = \langle 0, 0 \rangle$.

Second Derivative Test

If (a,b) is a critical point of $f(x,y)$ and the second order partial derivatives are continuous in a region that contains (a,b). Let

$$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2.$$

Then

(1) If $D > 0$ and $f_{xx}(a,b) > 0$, then (a,b) is a local minimum.

(2) If $D > 0$ and $f_{xx}(a,b) < 0$, then (a,b) is a local maximum.

(3) If $D < 0$, then (a,b) is a saddle point.

(4) If $D = 0$, this test is inconclusive.

Absolute Maximum and Minimum

Definition: A function f of two variables has an **absolute maximum** at (a, b) if $f(x, y) \leq f(a, b)$ for every (x, y) in the domain of f.

If $f(x, y) \geq f(a, b)$ for every point (x, y) in the domain of f, then f has an **absolute minimum** at (a, b).

Boundary points, closed sets and bounded sets

Definition: A point (a, b) in a subset D of $\mathbb{R}^2$ is a **boundary point** of D such that every disk with center (a, b) contains some points in D and points not in D.

Definition: A **closed set** in $\mathbb{R}^2$ is one that contains all its boundary points.

Definition: A **bounded set** in $\mathbb{R}^2$ is on that is contained in some disk.

Extreme value theorem for functions in two variables

If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and and absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

Finding absolute max and min values

D is closed, bounded set and f is continuous.

(1) Find the values of f at the critical points of f in D.

(2) Find the extreme values of f on the boundary of D.

(3) The largest of the values from step 1 and 2 is the absolute maximum value and the smallest of these values is the absolute minimum value.