Functions in several variables and limits

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Functions

Any function has three features:

- A **domain** set X;
- A **range or codomain** set Y;
- A **rule of assignment** - a rule that assign to each element x in X of the domain a "unique" element $f(x)$ in Y (the codomain).

The Range or Codomain of a function

Definition: The **Image** of a function $f : X \to Y$ is the set of elements of Y that are actual values of f.

$$\text{Image}(f) = \{y \in Y \mid y = f(x) \text{ for some } x \in X\}.$$

Scalar-valued functions

Scalar valued functions are functions such that the domain is $X \subseteq \mathbb{R}^n$ and the range (or codomain) is $\mathbb{R}$ or a subset of $\mathbb{R}$.

Vector-valued functions

Vector valued functions are functions such that the domain is $X \subseteq \mathbb{R}^n$ and the range (or codomain) is $\mathbb{R}^m$ or a subset of $\mathbb{R}^m$.

One-to-one and Onto

Definition: A function $f : X \to Y$ is **onto** (or **surjective**) if every element of Y is the image of some element of X, that is

$$\text{Image}(f) = Y$$

Definition: A function $f : X \to Y$ is called **one-to-one** (or **injective**) if no two distinct elements of the domain have the same image under f. That is, f is one-to-one if for any two $a, b \in X$ with $a \neq b$ then $f(a) \neq f(b)$.

The Graph of a function

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a scalar valued function. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, then the **graph** of f is:

$$\text{Graph} f = \{(\mathbf{x}, f(\mathbf{x})) \,|\, \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n\}$$

For example if $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, then the graph of f is the set of points in $\mathbb{R}^3$ that look like $(x, y, f(x, y))$, where (x, y) is in $\mathbb{R}^2$.

Level Curves

Let f be a function of two variables and let c be a constant. The set of all (x, y) in the plane such that $f(x, y) = c$ is called a **level curve** of f with value c.

Definition of limit

Definition: Let $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ be a function. Then

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x}) = L$$

means that given $\epsilon > 0$, you can find a $\delta > 0$ (often depending on ϵ) such that if $\mathbf{x} \in X$ and $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$, then $|f(\mathbf{x}) - L| < \epsilon$

Properties of limits

1. If $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ and $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = M$ then $\lim_{\mathbf{x}\to\mathbf{a}} (f+g)(\mathbf{x}) = L + M$.

2. If $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$, then $\lim_{\mathbf{x}\to\mathbf{a}} kf(\mathbf{x}) = kL$, where k is a scalar.

3. if $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ and $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = M$ then $\lim_{\mathbf{x}\to\mathbf{a}} (fg)(\mathbf{x}) = LM$

4. If $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ and $g(\mathbf{x}) \neq 0$ for $\mathbf{x} \in X$, and $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = M \neq 0$, then $\lim_{\mathbf{x}\to\mathbf{a}} (\frac{f}{g})(\mathbf{x}) = \frac{L}{M}$.

Continuous Functions

Definition: Let $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and let $\mathbf{a} \in X$. Then, f is **continuous at a** if

$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a}).$$

$\mathbf{f}$ is called **continuous** if it is continuous at every point of the domain X.

- The sum $f + g$ of two continuous functions is a continuous function.

- The scalar multiple of a continuous function kf is continous.

- The product fg and the quotient f/g (when defined) of two continuous functions is continuous.