Vector-Valued Functions

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Vector-valued functions

Let I be a subset of $\mathbf{R}$. Then a **vector-valued function** is a rule $\mathbf{r}$ that assigns to every real number t in I a unique vector in $\mathbb{R}^n$.

$$\mathbf{r}(t) = \langle x_1(t), x_2(t), \ldots, x_n(t) \rangle$$

For example, for $n = 3$:

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

NOTE: Domain is a subset of real numbers and range is $\mathbb{R}^3$.

Limits and continuity of vector-valued functions

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \mathbf{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$$

provided the limit of the component functions exist.

Definition: A vector-valued function $\mathbf{r}$ is **continuous at** a if

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$$

Definition of Derivative

The **derivative** $\mathbf{r}'$ is defined by

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

The vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve defined by $\mathbf{r}$.

Computing the derivative

Theorem: If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle\rangle$, where f, g and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$$

Differentiation Rules

Theorem: Suppose that $\mathbf{r}$ and $\mathbf{s}$ are differentiable vector functions, c a scalar, and f a real-valued function.

1. $\frac{d}{dt}[\mathbf{r}(t) + \mathbf{s}(t)] = \mathbf{r}'(t) + \mathbf{s}'(t)$
2. $\frac{d}{dt}[c\mathbf{r}(t)] = c\,\mathbf{r}'(t)$
3. $\frac{d}{dt}[f(t)\mathbf{r}(t)] = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$
4. $\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{s}(t)] = \mathbf{r}'(t) \cdot \mathbf{s}(t) + \mathbf{r}(t) \cdot \mathbf{s}'(t)$
5. $\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{s}(t)] = \mathbf{r}'(t) \times \mathbf{s}(t) + \mathbf{r}(t) \times \mathbf{s}'(t)$
6. $\frac{d}{dt}[\mathbf{r}(f(t))] = f'(t)\mathbf{r}'(f(t))$

Definite Integrals

The **definite integral** of a continuous vector function $\mathbf{r}(t)$ can be computed using

$$\int_a^b \mathbf{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$$

Definition of a path

Let $I = [a, b]$ be a closed interval for some numbers $a < b$. $I \subseteq \mathbb{R}$.

Definition: A **path in $\mathbb{R}^n$** is a continuous function $\mathbf{r} : I \longrightarrow \mathbb{R}^n$ where $\mathbf{r}(a)$ and $\mathbf{r}(b)$ are the **endpoints** of the path $\mathbf{r}$.

Velocity, speed and acceleration

Let $\mathbf{r} : I \longrightarrow \mathbb{R}^n$ be a differentiable path. Then

- The **velocity** $\mathbf{v}(t) = \mathbf{r}'(t)$.
- The **speed** is $\|\mathbf{v}(t)\|$.
- The **acceleration** is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

The tangent line

Let $\mathbf{r} : I \longrightarrow \mathbb{R}^n$ be a path and $\mathbf{v}(t_0) \neq \mathbf{0}$. Then the parametric equation of the tangent line at t_0 to the path $\mathbf{r}$ is

$$\mathbf{l}(t) = \mathbf{r}(t_0) + (t - t_0)\mathbf{v}_0.$$

Length of a path

Definition: The **length** $L(\mathbf{r})$ of a differentiable path $\mathbf{r} : [a, b] \to \mathbb{R}^n$ is the integral of its speed

$$L(\mathbf{r}) = \int_a^b \|\mathbf{r}'(t)\| \; dt$$