Power Series

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Definition of a power series

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

where x is a variable and c_n's are constants called **coefficients**.

A **power series about a** is a series of the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where a is a constant.

Theorem About Convergence of Power Series

Theorem: For a power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n,$$

there are three possibilities:

1. The series converges only when $x = a$.

2. The series converges for all x.

3. There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x - a| > R$.

Radius of Convergence

The number R in part (3) of the theorem is called **radius of convergence** . By convention if the series only converges at $x = a$, then we say the radius of convergence is $R{-}0$ and if the series converges for all x, we say the radius of convergence is $R = \infty$.

The **interval of convergence** is either
1. $[a - R, a + R]$ or
2. $[a - R, a + R)$ or
3. $(a - R, a + R]$ or
4. $(a - R, a + R)$.

Differentiation and Integration of Power Series

Theorem: If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ with radius of convergence $R > 0$, then f is differentiable (hence continuous) in the interval $(a-R, a+R)$ and

$$\begin{aligned}
f'(x) &= \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} \\
&= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots . \\
\int f(x)\,dx &= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \\
&= C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots
\end{aligned}$$