The Integral Test

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The Integral Test

Suppose

(1) f is a continuous, positive, decreasing function on $[1, \infty)$ and

(2) let $a_n = f(n)$.

Then the series $\sum_{n=1}^{\infty} a_n$ is convergent

if and only if

the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent.

Integral Test and Divergence/Convergence of series

Assume conditions (1) and (2) of Integral Test.

- If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum\limits_{n=1}^{\infty} a_n$ is convergent.
- If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

The p-series

Definition: Let p be a real number. Then the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called the p-**series.**

Theorem: The p-series is convergent if $p > 1$ and divergent if $p \leq 1$.

Remainder Estimate for the Integral Test

The **remainder**, R_n, is defined by

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots .$$

Remainder Estimate:
Suppose $f(k) = a_k$, where f is continuous, positive and decreasing function for $x \geq n$ and $\sum a_n$ is convergent.
If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

Estimate for the sum of a series

Let s be the sum of a series and s_n be the n-th partial sum: If we can use the integral test to test for convergence, then

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \le s \le s_n + \int_{n}^{\infty} f(x)\,dx$$