Math 13 — W 2000 —Handout 3
Differentiability, the derivative map, and the derivative matrix

Definition. A function $f:\mathbf{R}^n \to \mathbf{R}^m$ is *differentiable at the point* $\mathbf{x}_0 \in \mathbf{R}^n$ if there is a linear transformation $L:\mathbf{R}^n \to \mathbf{R}^m$ such that $f(\mathbf{x}_0+\mathbf{h})$ is given, to first order, by $f(\mathbf{x}_0+\mathbf{h}) = f(\mathbf{x}_0)+L(\mathbf{h})$; that is, if $f(\mathbf{x}_0+\mathbf{h})-f(\mathbf{x}_0)-L(\mathbf{h})$ goes to zero as $\mathbf{h}\to\mathbf{0}$, but *faster* than $\mathbf{h}$ itself does:

$$(*) \qquad \lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{x}_0+\mathbf{h})-f(\mathbf{x}_0)-L(\mathbf{h})}{\|\mathbf{h}\|} = \mathbf{0}.$$

If such a linear transformation L exists, we call it the *derivative of f at* $\mathbf{x}_0$, and denote it $Df(\mathbf{x}_0)$. Thus $Df(\mathbf{x}_0)$ is a linear transformation $\mathbf{R}^n \xrightarrow{Df(\mathbf{x}_0)} \mathbf{R}^m$ which serves as the *linear part* of the first-order approximation to f near $\mathbf{x}_0$; indeed, the first-order (affine-linear) approximation to f near $\mathbf{x}_0$ is given by

$$f(\mathbf{x}_0+\mathbf{h}) \approx f(\mathbf{x}_0)+(Df(\mathbf{x}_0))(\mathbf{h}),$$

that is, $f(\mathbf{x}_0+\mathbf{h}) = f(\mathbf{x}_0)+(Df(\mathbf{x}_0))(\mathbf{h})+$(terms which are "quadratic or higher" in $\mathbf{h}$) — this is the beginning of a "Taylor series" in higher dimensions.

Note: If we write $\mathbf{x}=\mathbf{x}_0+\mathbf{h}$, so that $\mathbf{h}=\mathbf{x}-\mathbf{x}_0$, we can rewrite

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{x}_0+\mathbf{h})-f(\mathbf{x}_0)-(Df(\mathbf{x}_0))(\mathbf{h})}{\|\mathbf{h}\|} = \mathbf{0}$$

as

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{f(\mathbf{x})-f(\mathbf{x}_0)-(Df(\mathbf{x}_0))(\mathbf{x}-\mathbf{x}_0)}{\|\mathbf{x}-\mathbf{x}_0\|} = \mathbf{0},$$

thereby recovering the definition on page 129 of Marsden/Tromba/Weinstein.

Assuming that $Df(\mathbf{x}_0)$ exists, it is a linear transformation, so it has a representing matrix $[Df(\mathbf{x}_0)]$. How do we find this representing matrix?

Recall that for *any* linear transformation $L:\mathbf{R}^n \to \mathbf{R}^m$, the representing matrix $[L]$ is the $m\times n$ matrix whose columns are the vectors $L(\mathbf{e}_1)$, $L(\mathbf{e}_2),\ldots,L(\mathbf{e}_n)$, where $\mathbf{e}_j=(0,0,\ldots,0,1,0,\ldots,0)$ is the jth standard coordinate basis vector. Thus, to determine $[Df(\mathbf{x}_0)]$, we should compute its columns $(Df(\mathbf{x}_0))(\mathbf{e}_j)$ for $j=1,2,\ldots,n$. Now f is a function $f:\mathbf{R}^n\to\mathbf{R}^m$, so it has m component functions $f_1, f_2,\ldots,f_m$, i.e.,

$f(x_1, x_2, \ldots, x_n) = (f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_m(x_1, x_2, \ldots, x_n))$, where each f_j is a real-valued function $f_j : \mathbf{R}^n \to \mathbf{R}$. We know by definition (equation (*)) that

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) - (Df(\mathbf{x}_0))(\mathbf{h})}{\|\mathbf{h}\|} = \mathbf{0}.$$

Let's consider the special case when we take the limit by restricting $\mathbf{h}$ to be of the form $\mathbf{h} = t\mathbf{e}_j$ for a real number t. That is, we will let $\mathbf{h}$ approach zero in the $\mathbf{e}_j$-direction by letting t approach zero:

[since $Df(x_0)$ is a linear function, $Df(x_0)(t\vec{e}_j) = t\,Df(x_0)(\vec{e}_j)$]

$$\mathbf{0} = \lim_{t \to 0} \frac{f(\mathbf{x}_0 + t\mathbf{e}_j) - f(\mathbf{x}_0) - (Df(\mathbf{x}_0))(t\mathbf{e}_j)}{\|t\mathbf{e}_j\|} = \lim_{t \to 0} \frac{f(\mathbf{x}_0 + t\mathbf{e}_j) - f(\mathbf{x}_0) - t(Df(\mathbf{x}_0))(\mathbf{e}_j)}{|t|}$$

$$= \lim_{t \to 0} \frac{f(\mathbf{x}_0 + t\mathbf{e}_j) - f(\mathbf{x}_0) - t(Df(\mathbf{x}_0))(\mathbf{e}_j)}{t} = \lim_{t \to 0} \left(\frac{f(\mathbf{x}_0 + t\mathbf{e}_j) - f(\mathbf{x}_0)}{t} - (Df(\mathbf{x}_0))(\mathbf{e}_j) \right)$$

which says that

(**) $$\boxed{(Df(\mathbf{x}_0))(\mathbf{e}_j) = \lim_{t \to 0} \frac{f(\mathbf{x}_0 + t\mathbf{e}_j) - f(\mathbf{x}_0)}{t}.}$$

We can recognize the limit in (**) as something familiar. Let $\mathbf{x}_0 = (a_1, a_2, \ldots, a_n)$. Remembering that $f(\mathbf{x}_0 + t\mathbf{e}_j) - f(\mathbf{x}_0)$ is an element of $\mathbf{R}^m$,

[jth place]

$$\frac{f(\mathbf{x}_0 + t\mathbf{e}_j) - f(\mathbf{x}_0)}{t} = \frac{1}{t}\big(f((a_1, a_2, \ldots, a_n) + t(0, \ldots, 0, 1, 0, \ldots, 0)) - f(a_1, a_2, \ldots, a_n)\big) =$$

[jth place]

$$\frac{1}{t}\big(f(a_1, a_2, \ldots, a_j + t, \ldots, a_n) - f(a_1, a_2, \ldots, a_n)\big) =$$

$$\frac{1}{t}\begin{bmatrix} f_1(a_1, a_2, \ldots, a_j + t, \ldots, a_n) - f_1(a_1, a_2, \ldots, a_n) \\ f_2(a_1, a_2, \ldots, a_j + t, \ldots, a_n) - f_2(a_1, a_2, \ldots, a_n) \\ \vdots \\ f_m(a_1, a_2, \ldots, a_j + t, \ldots, a_n) - f_m(a_1, a_2, \ldots, a_n) \end{bmatrix} =$$

$$\begin{bmatrix} \frac{f_1(a_1, a_2, \ldots, a_j + t, \ldots, a_n) - f_1(a_1, a_2, \ldots, a_n)}{t} \\ \frac{f_2(a_1, a_2, \ldots, a_j + t, \ldots, a_n) - f_2(a_1, a_2, \ldots, a_n)}{t} \\ \vdots \\ \frac{f_m(a_1, a_2, \ldots, a_j + t, \ldots, a_n) - f_m(a_1, a_2, \ldots, a_n)}{t} \end{bmatrix}, \text{ so}$$

$$(Df(\mathbf{x}_0))(\mathbf{e}_j) = \lim_{t \to 0} \frac{f(\mathbf{x}_0 + t\mathbf{e}_j) - f(\mathbf{x}_0)}{t} =$$

$$\lim_{t\to 0}\begin{bmatrix} \dfrac{f_1(a_1,a_2,\ldots,a_j+t,\ldots,a_n)-f_1(a_1,a_2,\ldots,a_n)}{t} \\ \dfrac{f_2(a_1,a_2,\ldots,a_j+t,\ldots,a_n)-f_2(a_1,a_2,\ldots,a_n)}{t} \\ \vdots \\ \dfrac{f_m(a_1,a_2,\ldots,a_j+t,\ldots,a_n)-f_m(a_1,a_2,\ldots,a_n)}{t} \end{bmatrix} =$$

$$\begin{bmatrix} \lim_{t\to 0}\dfrac{f_1(a_1,a_2,\ldots,a_j+t,\ldots,a_n)-f_1(a_1,a_2,\ldots,a_n)}{t} \\ \lim_{t\to 0}\dfrac{f_2(a_1,a_2,\ldots,a_j+t,\ldots,a_n)-f_2(a_1,a_2,\ldots,a_n)}{t} \\ \vdots \\ \lim_{t\to 0}\dfrac{f_m(a_1,a_2,\ldots,a_j+t,\ldots,a_n)-f_m(a_1,a_2,\ldots,a_n)}{t} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_j}(a_1,a_2,\ldots,a_n) \\ \dfrac{\partial f_2}{\partial x_j}(a_1,a_2,\ldots,a_n) \\ \vdots \\ \dfrac{\partial f_m}{\partial x_j}(a_1,a_2,\ldots,a_n) \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_j}(\mathbf{x}_0) \\ \dfrac{\partial f_2}{\partial x_j}(\mathbf{x}_0) \\ \vdots \\ \dfrac{\partial f_m}{\partial x_j}(\mathbf{x}_0) \end{bmatrix}.$$

Thus the jth column of the representing matrix $[Df(\mathbf{x}_0)]$ is $\begin{bmatrix} \dfrac{\partial f_1}{\partial x_j}(\mathbf{x}_0) \\ \dfrac{\partial f_2}{\partial x_j}(\mathbf{x}_0) \\ \vdots \\ \dfrac{\partial f_m}{\partial x_j}(\mathbf{x}_0) \end{bmatrix}$. This means that the representing matrix $[Df(\mathbf{x}_0)]$ of the linear transformation $Df(\mathbf{x}_0)$ is given by

(***)
$$[Df(\mathbf{x}_0)] = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1}(\mathbf{x}_0) & \dfrac{\partial f_1}{\partial x_2}(\mathbf{x}_0) & \cdots & \dfrac{\partial f_1}{\partial x_n}(\mathbf{x}_0) \\ \dfrac{\partial f_2}{\partial x_1}(\mathbf{x}_0) & \dfrac{\partial f_2}{\partial x_2}(\mathbf{x}_0) & \cdots & \dfrac{\partial f_2}{\partial x_n}(\mathbf{x}_0) \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial f_m}{\partial x_1}(\mathbf{x}_0) & \dfrac{\partial f_m}{\partial x_2}(\mathbf{x}_0) & \cdots & \dfrac{\partial f_m}{\partial x_n}(\mathbf{x}_0) \end{bmatrix}.$$

Generally we will dispense with the cumbersome notation $[Df(\mathbf{x}_0)]$ and will refer to both the linear transformation $Df(\mathbf{x}_0):\mathbf{R}^n \to \mathbf{R}^m$ and its representing matrix $[Df(\mathbf{x}_0)]$ by the *same* symbol, $Df(\mathbf{x}_0)$. Since knowing a linear transformation is equivalent to knowing its representing matrix, this slight abuse of notation will cause no confusion.

Special case: If $m = 1$, so that $f:\mathbf{R}^n \to \mathbf{R}$ is an ordinary real-valued function, then (***) reduces to

$[Df(\mathbf{x}_0)] = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}_0) & \frac{\partial f}{\partial x_2}(\mathbf{x}_0) & \cdots & \frac{\partial f}{\partial x_n}(\mathbf{x}_0) \end{bmatrix}$, which we recognize as the gradient

$\nabla f(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}_0) \\ \frac{\partial f}{\partial x_2}(\mathbf{x}_0) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}_0) \end{bmatrix}$ rewritten as a row vector. In particular, for a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbf{R}^n$,

we have $(Df(\mathbf{x}_0))(\mathbf{v}) = [Df(\mathbf{x}_0)]\,\mathbf{v} = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}_0) & \frac{\partial f}{\partial x_2}(\mathbf{x}_0) & \cdots & \frac{\partial f}{\partial x_n}(\mathbf{x}_0) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} =$

$\frac{\partial f}{\partial x_1}(\mathbf{x}_0)v_1 + \frac{\partial f}{\partial x_2}(\mathbf{x}_0)v_2 + \ldots + \frac{\partial f}{\partial x_n}(\mathbf{x}_0)v_n = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}_0) \\ \frac{\partial f}{\partial x_2}(\mathbf{x}_0) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}_0) \end{bmatrix} \bullet \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \nabla f(\mathbf{x}_0) \bullet \mathbf{v}$, the directional

derivative of f in the $\mathbf{v}$-direction. Indeed, equation (**) says exactly that for a function $f : \mathbf{R}^n \to \mathbf{R}^m$, $(Df(\mathbf{x}_0))(\mathbf{e}_j)$ is a vector-valued generalization of the usual jth partial derivative. As we will show in homework, $(Df(\mathbf{x}_0))(\mathbf{v})$ is a vector-vauled generalization of the directional derivative in the $\mathbf{v}$-direction:

$$\boxed{(Df(\mathbf{x}_0))(\mathbf{v}) = \lim_{t \to 0} \frac{f(\mathbf{x}_0 + t\mathbf{v}) - f(\mathbf{x}_0)}{t}.}$$

This gives a nice geometric way of thinking about the derivative as a linear map.