PROJECT 1: INVARIANT AREAS

MATH 9 FALL 2001

Directions: For this project, you may work in groups of up to three individuals. If you choose, you may of course work alone or with one other person. The project is due on **Wednesday October 17, 2001** at the beginning of class. No late projects will be accepted. One write up is due from each group and each group member's name should be clearly marked on the paper that is turned in. When writing up the solution, use complete sentences and detailed explanations of each step of your solution. Another student in the course should be able to read your solution and understand every step.
For this project, the honor principle has the following interpretation: you are allowed to use any materials you want, books, notes, etc. and you are allowed to generally discuss the project with the tutors for the class and the instructor. However, you must give your own explanation of the answers to the questions which reflect *your own understanding* of the answer. In other words, if asked, each member of the group should be able to reproduce and explain the answers that the group turns in. Any deviation from these directions will be regard as a violation of the Honor Principle.

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Part 1: Consider the piece of the hyperbola $xy = 1$ with $x > 0$ and $y > 0$. We are going to show that the lines tangent to theis curve have a very special property.

Definition: *The line tangent to the curve with equation* $y = f(x)$ *at a point* $(a, f(a))$ *on the curve is the line through* $(a, f(a))$ *with slope* $f'(a)$*. We call this line a tangent line to the curve at the given point.*

The tangent line at any point of this piece of the hyperbola forms a right triangle with the coordinate axes. In this part, you need to show that the area of the triangle *does not depend* on the point chosen. To prove this, you may want to proceed as follows. Expressing y as a function of x, you are led to consider the function $H(x) = \frac{1}{x}$. What is the equation of the line tangent to the graph of $H(x)$ at $(a, H(a))$? Draw a graph of this function, a typical tangent line, and the triangle in question. What is the area of the triangle? If the area depends on the point $(a, H(a))$, you have made a mistake.

Part 2: This part is significantly harder than part 1. The goal of this part is to find *all* possible curves that have the same property, i.e. that the area triangle formed be the tangent line at a point and the coordinate axes does not depend on the point chosen. Follow the outline below:

(1) Setup: Let f be the function such that $f(x) > 0$ whenever $x > 0$, the derivative $f'(x)$ is always negative and $f''(x)$ is never zero. Consider the triangle formed by the coordinate axes and the line tangent to the graph of f at a point $(a, f(a)$.
(2) Assume that the area is independent of $(a, f(a))$, i.e. it is constant. This gives you an equation in the variable a. Differentiate it in a, simplify, factor and use the conditions on f to isolate a simple differential equation which characterizes f.
(3) Solve the differnetial equation to find all possible curves with the "invariant area" property.