ERROR ESTIMATES IN TAYLOR APPROXIMATIONS

Suppose we approximate a function $f(x)$ near $x = a$ by its Taylor polynomial $T_n(x)$. How accurate is the approximation? In other words, how big is the error $f(x) - T_n(x)$? This error is often called the remainder $R_n(x)$ since it's what's left if we replace $f(x)$ by $T_n(x)$.

Examples. Since $T_0(x) = f(a)$, we have $R_0(x) = f(x) - f(a)$. Since $T_1(x) = f(a) + f'(a)(x - a)$ is the tangent line to f at a, the remainder $R_1(x)$ is the difference between $f(x)$ and the tangent line approximation of f.

An important point:

- You can almost never find the exact value of $R_n(x)$. If you knew the value exactly, then you would know the precise value of $f(x)$ (since it's easy to compute $T_n(x)$ exactly and since $f(x) = T_n(x) + R_n(x)$). Generally you're using the Taylor approximation because it's not possible to find the value exactly! So the best we can hope to do is get an upper bound on the size $|R_n(x)|$ of the error.

The following formula gives us a way of bounding the error $R_n(x)$.

Lagrange's formula. There is some number c between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1}.$$

(We don't know what c is! We only know there is such a c.)

How do we use Lagrange's formula to get a bound on $|R_n(x)|$?

> If you can find a positive real number M such that $|f^{(n+1)}(c)| \leq M$ *for all* c between a and the point x of interest, then Lagrange's formula tells you that
>
> (1) $$|R_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1}.$$

Note. Frequently, it's too hard to find the *exact* maximum of $|f^{(n+1)}(c)|$ on the interval between a and x. Instead, you can look for a number M that you know is at least as big as the maximum (so you *over*estimate the maximum). For example, if $f^{n+1}(c)$ is $\sin(c)$ or $\cos(c)$, then you can safely use the upper bound $M = 1$, even if the interval doesn't include any points where the value of $\sin$ or $\cos$ is actually equal to 1. Similarly, if $|f^{(n+1)}(c)| = \sqrt{c}$ and the interval of interest is, say, [1,3], then the actual maximum is $\sqrt{3}$, which is rather ugly, but you could use the upper bound $M = 2$ since that's bigger than the maximum. You just need to be careful not to underestimate the maximum. In this example, you couldn't use $M = 1$, for example, since that's smaller than the actual maximum $\sqrt{3}$.

Example. Suppose we want to approximate the value of e, say to within an error of at most 0.001. Since $e = e^1$, we could use a suitable Taylor polynomial for the function $f(x) = e^x$ to estimate e^1. What should we use for our basepoint? The one value we know exactly is $f(0) = e^0 = 1$. So we will use a Taylor polynomial $T_n(x)$ for e^x about $a = 0$. We can then estimate e by computing $T_n(1)$. What's the smallest degree Taylor polynomial we can use to get the guaranteed accuracy? (I.e., what is the smallest n we can use?)

Letting $f(x) = e^x$, we have $f'(x) = e^x$ and, in fact, $f^{(n)}(x) = e^x$ for all n. So

$$f(0) = 1 = f'(0) = f''(0) = f'''(0) = \dots$$

Thus, e.g.,

$$T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$$

and, more generally,

$$T_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

We are interested in the error $R_n(1)$ in approximating e^1 by $T_n(1)$.

We use Equation (2). Here M is an upper bound for $|f^{(n+1)}(c)| = |e^c|$ for c between 0 and 1. Here's some things we know:

- We know e^c is positive, so $|e^c| = e^c$.
- e^x is an increasing function, so it's biggest value on the interval $[0, 1]$ occurs at the righthand endpoint 1.
- We don't know the exact value of $e = e^1$ (that's what we're trying to approximate!), but we do know that $e^1 < 3$. (You've probably heard that it's around 2.7.)

So the maximum of e^c for $c \in [0, 1]$ is e^1, which is less than 3. So we are safe using $M = 3$ in Equation (2). (We could also use, say, 2.8 since it's also bigger than any value of e^c for $c \in [0, 1]$, but 3 is reasonable and easy to work with.)

Next since $x = 1$ and $a = 0$, we have $|x - a|^{n+1} = 1$, so Equation (2) yields

$$|R_n(1)| \le \frac{3}{(n+1)!}.$$

So we need to find the smallest n such that

$$\frac{3}{(n+1)!} \le 0.001 = \frac{1}{1000}.$$

Let's experiment a bit. If we, say, start with $n = 5$, we get
$$\frac{3}{(5+1)!} = \frac{3}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{1}{6\cdot 5\cdot 4\cdot 2\cdot 1} = \frac{1}{240}.$$
That's not quite good enough, so let's try $n = 6$.
$$\frac{3}{(6+1)!} = \frac{3}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{1}{7\cdot 6\cdot 5\cdot 4\cdot 2\cdot 1} = \frac{1}{1680} < \frac{1}{1000}.$$
So $n = 6$ works! This yields
$$e = e^1 \sim T_6(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!}$$
to within 0.001.

Often, one wants to to approximate a function on an interval about the basepoint a, say on an interval $|x - a| \le d$. We can then use the formula in the box above in the following way:

Taylor's Inequality. If you can find a positive real number M such that $|f^{(n+1)}(x)| \le M$ for all x such that $|x - a| \le d$, then

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \tag{2}$$

for all x in the interval $|x - a| \le d$.

Example. Suppose we use the first degree Taylor polynomial (i.e., the tangent line approximation) about $a = 9$ to estimate $f(x) = \sqrt{x}$ on the interval $[8.5, 9.5]$. Show that the magnitude of the error is less than 0.01.

We need to show that $|R_1(x)| \le 0.01$ when $|x - 9| \le 0.5$ (since [8.5,9.5] is the interval $|x - 9| \le 0.5$.) We use Taylor's inequality with $n = 1$. We have $f'(x) = \frac{1}{2}x^{-1/2}$ and $f''(x) = \frac{-1}{4}x^{-3/2} = -\frac{1}{4x^{3/2}}$. We first need an upper bound M for $|f''(x)| = \frac{1}{4x^{3/2}}$ on [8.5,9.5]. The largest value occurs when the denominator is the smallest, so the actual maximum is $\frac{1}{4(8.5)^{3/2}}$. This is our best choice of M but is rather ugly. We'll simplify things in a bit.

By Taylor's inequality, we have
$$|R_1(x)| \le \frac{M}{2!}|x-9|^2 \le \frac{M}{2!}(0.5)^2 = \frac{M}{8}$$
when $|x - 9| \le 0.5$. So we just need to know whether $\frac{M}{8} \le 0.01$, i.e., whether $M \le 0.08 = \frac{8}{100} = \frac{2}{25}$. Well, asking whether $\frac{1}{4(8.5)^{3/2}} \le \frac{2}{25}$ is the same as asking whether $4(8.5)^{3/2} \ge \frac{25}{2}$. This is true since $(8.5)^{3/2}$ is certainly bigger than 8, so $4(8.5)^{3/2} > 4(8) = 32 > \frac{25}{2}$. So we have verified that our error is well under 0.01.