MATH 8
WEEKLY HW #2

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Introduction

Remember that the weekly homework problems are a chance to engage with the material on a deeper level than simply applying the algorithms presented in the textbook. Often, you might need to do some experimenting with the problem statements before being able to make some progress and it might not be obvious at first how to begin the problem. This is normal and can be a valuable part of the learning process. Additionally, if you are stuck, that is a great time to attend tutorial or office hours to talk about the problem. There are many ways to answer all of these problems correctly and the solutions in this document are simply meant to be a guide for one set of possible approaches.

1. Use Taylor polynomials to compute the integral below to at least 4 digits of accuracy:

$$\int_0^1 e^{x^2}dx$$

Perhaps the nicest way to approach this problem is to start by playing with the Taylor polynomials for e^{x^2}. They take a particularly nice form:

$$T_n(x) = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24} + \cdots$$

with the individual terms given by: $\frac{x^{2n}}{n!}$. Integrating these polynomials term by term is also straightforward since:

$$\int_0^1 \frac{x^{2n}}{n!}dx = \frac{x^{2n+1}}{(2n+1)\cdot n!}\Big|_0^1 = \frac{1}{(2n+1)\cdot n!}$$

The integral of the function is just the sum of the integrals of the individual terms:

$$\int_0^1 T_n(x)dx = \sum_{k=0}^{n} \frac{1}{(2k+1)\cdot k!}$$

and it is then just a matter of calculating this sum for a few reasonable values of n. For example, taking $n = 10$ gives us a sum of $\approx 1.46265...$ which actually matches the first eight digits of the integral that we are interested in.

You could alternatively use the Taylor Error bound to get a solution to this problem but it is a little more involved.

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2. Estimate $e^{.1}$ and $e^{-.1}$ to 4 digits of accuracy.

For this problem we can just use the Taylor polynomial for e^x centered at $a = 0$ with $d = .1$. Since e^x is increasing on this interval we can take $M = e^{.1}$ and we know that we want the error to be less than .0001. Substituting this in to our formula we get:

$$\frac{e^{.1}}{(n+1)!}.1^{n+1} < .0001$$

Looking at a couple of small values of n shows that taking $n = 3$ is plenty of terms since $\frac{e^{.1}}{(4)!}.1^4 < .00001$. Computing the 3rd order taylor polynomial gives:

$$T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

and substituting in our values gives the final answers:

$$T_3(.1) \approx 1.105$$

$$T_3(-.1) \approx .9048$$

Which are quite close to the actual values as desired.

3. Express $3.\overline{142857}$ as a ratio of integers

This is an application of the geometric series formula. We can rewrite the desired number as:

$$\begin{aligned}
3.\overline{142857} &= 3 + \frac{142857}{1000000} + \frac{142857}{1000000^2} + \cdots \\
&= 3 + \textstyle\sum_{n=1}^{\infty} \frac{142857}{1000000^n} \\
&= 3 + \frac{142857}{1000000}\left(\frac{1}{1 - \frac{1}{1000000}}\right) \\
&= 3 + \frac{142857}{999999} \\
&= \frac{22}{7}
\end{aligned}$$

4. Sequence Algebra

a. **If $\sum a_n$ and $\sum b_n$ are convergent series, can $\sum a_n + b_n$ diverge?**

This follows directly from Theorem 8(ii) in Section 11.2 of the textbook.

b. **If $\sum a_n$ and $\sum b_n$ are divergent series, can $\sum a_n + b_n$ converge?**

Although we might be tempted by the easy answer like the previous problem the situation here is actually a little more complex. The problem is that we might have cancellation of terms. For example if $a_n = 1$ and $b_n = -1$ then both original series definitely diverge but their sum $a_n + b_n = 0$ and we can add up as many zeros as we like without any trouble.

In some sense this is the only thing that can go wrong, since if a_n and b_n both have all positive terms then their sum must diverge if they both individually do.

c. **If $\sum a_n$ is a convergent series and $f(x) = cx + d$ for some real constants c and d, for what values of c and d does $\sum f(a_n)$ converge?** We are looking at a series of the form $\sum_{n=1}^{\infty} ca_n + d$. By Theorem 8(i) we know that $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$ converges since $\sum a_n$ does. This seems to suggest that any value of c might work for us. On the other hand d seems to cause some problems. For any value of d that is not zero the series $\sum_{n=1}^{\infty} d$ diverges to infinity. Thus, for there to be any hope of $\sum f(a_n)$ to converge we need to have $d = 0$. When $d = 0$ we can apply Theorem 8(i) directly to $\sum f(a_n)$ and hence any value of c is permissible.

d. **If $\sum a_n$ is a convergent series with $1 \geq a_n \geq 0$ does $\sum a_n \cdot a_{n+1}$ converge?** The key idea here is that $a_n \cdot a_{n+1} \leq a_n$ since all the terms of a_n are between zero and one. Thus, we can apply the comparison test to $\sum a_n \cdot a_{n+1} \leq \sum_{n=1}^{\infty} a_n$ to see that it converges.

e. **For what values of p does $\sum n^p \cdot \ln(n)$ converge?** This setup is reminiscent of the way we analyzed p–series at the beginning which suggests that we might want to use the integral test. We can integrate using parts to get a limit expression in terms of p:

$$\int_1^{\infty} x^p \ln(x) dx = \quad \lim_{t\to\infty} \frac{x^{p+1}\ln(x)}{p+1} - \frac{x^{p+1}}{(p+1)^2}\Big|_1^t$$

Looking at this limit we see that we need $p + 1 < 0$ to move the polynomial terms to the denominator so the limit can converge. This is equivalent to taking $p < -1$ and for any such value of p the integral converges and hence by the integral test, the series converges.

f. **For what values of p does $\sum(n \cdot (\ln(n))^p)^{-1}$ converge?**

This problem is similar to the previous one in that we can integrate to get a limit formula that determines p. In this case, we want to use integration by substitution with $u = \ln(x)$ to get:

$$\int_1^{\infty} (x \cdot \ln(x)^p)^{-1} dx = \lim_{t\to\infty} \frac{\ln(x)^{1-p}}{1-p}\Big|_1^t$$

Again we see that we want to have $1 - p < 0$ or equivalently $p > 1$ in order for the integral and series to converge.

5. Does the series $\sum \frac{n!}{n^n}$ converge?

At this point we don't have very much experience with factorials so it isn't immediately clear from looking at the sum whether or not we should expect it to converge. If we check out the behavior of the series by adding up some partial sums we see that it seems to be converging to a value of ≈ 1.8 very rapidly and for even fairly small values of n (like $n = 6$ or $n = 7$) the terms of the sequence are already quite small ($\frac{6!}{6^6} \approx .015$ and $\frac{7!}{7^7} \approx .0061$) which certainly seems to be encouraging.

Let's take a look at the nth term of this sequence:

$$a_n = \frac{n!}{n^n} = \frac{n(n-1)(n-2)\cdots(3)(2)(1)}{n \cdot n \cdot n \cdots n} = \frac{n}{n}\frac{n-1}{n}\cdots\frac{2}{n}\frac{1}{n}$$

Each product consists of a n terms each of which is less than or equal to 1, multiplied together. This suggests that the total product is quite small for each a_n since each multiplcation shrinks the previous total a little bit. Since the terms are going to zero quickly, we have a lot of "wiggle room" to use for the comparison test.

There are lots of possible series that we could compare to in this problem but we will choose a particularly simple one:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

which converges by the p–series test. In order to compare our series to this one we can to make two comparisons.

First, we can check that $\dfrac{n!}{n^n} < \dfrac{n!}{(n+2)!}$ when $n \geq 6$. This is a case where checking the first couple of terms isn't enough since our inequality isn't true until $n = 6$. However, this is ok, since we can change finitely many terms at the beginning of our series without affecting the convergence. Thus, by the comparison test our original series converges if

$$\sum_{n=6}^{\infty} \frac{n!}{(n+2)!} = \sum_{n=6}^{\infty} \frac{1}{(n+2)(n+1} = \sum_{n=6}^{\infty} \frac{1}{n^2+3n+2}$$

converges. This final series looks like a more familar example of a comparison test and we know that making the denominator smaller makes the individual terms larger so we have:

$$\sum_{n=6}^{\infty} \frac{1}{n^2+3n+2} < \sum_{n=6}^{\infty} \frac{1}{n^2}$$

Since we know this series converges, by the comparison test our original series must also converge.

6. If $f(x)$ is a degree four polynomial and $g(x)$ is a degree 6 polynomial with all positive coefficients, does the series $\sum \frac{f(n)}{g(n)}$ converge?

Looking at this problem, we might guess that it should converge since the leading exponent of n in the denominator is two larger than the leading power in the numerator. Checking the convergence for a couple of choices of polynomials can provide more evidence that the series does converge (for example $\frac{(2n^4+3)}{(5n^6+4n^2+17)} \approx .45$).

To show that it does converge we can use the limit comparison test on two arbitrary polynomials with positive coefficients as follows. Let $f(x) = ax^4+bx^3+cx^2+dx+e$ and $g(x) = hx^6+ix^5+jx^4+kx^3+lx^2+mx+p$. Then we can compute the limit of the quotient of f and g with their leading terms:

$$\lim_{n\to\infty} \frac{\frac{f(n)}{g(n)}}{\frac{n^6}{n^4}} = \lim_{n\to\infty} \frac{f(n)}{g(n)}\frac{n^6}{n^4} = \lim_{n\to\infty} \frac{a+\frac{b}{n}+\frac{c}{n^2}+\frac{d}{n^3}+\frac{e}{n^4}}{h+\frac{i}{n}+\frac{j}{n^2}+\frac{k}{n^3}+\frac{l}{n^4}+\frac{m}{n^5}+\frac{p}{n^6}} = \frac{a}{h}$$

Since this limit converges we know that $\sum \frac{f(n)}{g(n)}$ converges if and only if $\sum \frac{1}{n^2}$ converges. Since it does we know that this series converges for any choice of coefficients for f and g.

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