1. Which of the following statements are always true? Write "**T**" for true and "**F**" for false. As a reminder, $n \in \mathbb{N}$ means that n is a non-negative integer. Your computations will not be graded on this problem.

(a) ☐ If $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$ then $\lim_{n\to\infty} a_n = L$.

(b) ☐ If $\lim_{n\to\infty} a_n = 0$ then $\lim_{n\to\infty} \frac{1}{a_n} = +\infty$.

(c) ☐ The series $\sum_{k=0}^{\infty} c_k x^k$ and $\sum_{k=1}^{\infty} k c_k x^{k-1}$ always have the same radius of convergence.

(d) ☐ The series $\sum_{k=0}^{\infty} c_k x^k$ and $\sum_{k=0}^{\infty} \frac{1}{k+1} c_k x^{k+1}$ always have the same interval of convergence.

(e) ☐ We can use the integral test to determine whether the series $\sum_{k=1}^{\infty} \frac{\sin k}{k}$ converges.

(f) ☐ If $\lim_{n\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L < 1$, then $\sum_{k=0}^{\infty} a_k$ converges absolutely.

(g) ☐ If $\lim_{n\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L > 1$, then $\sum_{k=0}^{\infty} a_k$ does not converge absolutely but might converge conditionally.

(h) ☐ Because the function $f(x) = \frac{1}{1+x^2}$ has no vertical asymptotes or points of discontinuity, the interval of convergence of its Taylor series must be $(-\infty, \infty)$.

(i) ☐ If $0 \le a_n \le b_n \le c_n$ for all n, and the sequences (a_n) and (c_n) both converge, then (b_n) must converge also.

(j) ☐ If (a_n) is an increasing sequence all of whose terms are negative, then it must converge.

(k) ☐ If the area under the curve $y = f(x)$ between $x = 1$ and $x = a$ is always between $\ln(a)$ and a, then $\int_0^{\infty} f(x)\,dx$ converges.

2. Find two ways of determining whether the series $\sum_{k=2}^{\infty} \frac{k}{k^2 - 1}$ converges.

   Way 1:

   Way 2:

3. Taylor's inequality says: If $|f^{(k+1)}(x)| \leq M$ for all x with $|x-a| \leq d$, then

$$|R_k(x)| \leq \frac{M}{(k+1)!}\,|x-a|^{k+1} \qquad \text{for all } x \text{ with } |x-a| \leq d.$$

Use Taylor's inequality to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2).$$

4. For each of the following 4 series, exlpain why the suggested test is applicable and determine whether the series converges.

$\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^4}$ (Integral Test)

$\displaystyle\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{n^2 - n}$ (Alternating Series Test)

$$\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{n^4+2n}} \text{ (Limit Comparison Test)}$$

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} \text{ (Ratio Test)}$$

5. Determine the interval of convergence of the power series $\sum_{n=0}^{\infty}(-3)^{n+1}(x-2)^n$.

   In case you want a challange: Which function has this series as its Taylor series at 2?

6. Find the Taylor series for
$$f(x) = \int_0^x \frac{dt}{1+t^4}.$$
What is the interval of convergence of your series?