**Worksheet #9**

Determine convergence or divergence of the series. Indicate which test you used.

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^{100}}$

**Solution:** We shall try the ratio test.

$$lr \lim_{n\to\infty} \left| \frac{(n+1)!}{(n+1)^{100}} \frac{n^{100}}{n!} \right| = \lim_{n\to\infty} \left| \frac{(n+1)\, n^{100}}{(n+1)^{100}} \right|$$
$$= \lim_{n\to\infty} \left| \frac{n^{100}}{(n+1)^{99}} \right| \to \infty$$

Thus the series diverges by the ratio test.

(2) $\displaystyle\sum_{k=1}^{\infty} \frac{3^k + k}{k!}$

**Solution:** Note $\displaystyle\sum_{k=1}^{\infty} \frac{3^k + k}{k!} = \sum_{k=1}^{\infty} \frac{3^k}{k!} + \sum_{k=1}^{\infty} \frac{k}{k!}$.

We shall apply the ratio test to both series.

$$\lim_{k\to\infty} \frac{3^{k+1}}{(k+1)!} \frac{k!}{3^k} = \lim_{k\to\infty} \frac{3}{k+1} = 0$$
$$\lim_{k\to\infty} \frac{k+1}{(k+1)!} \frac{k!}{k} = \lim_{k\to\infty} \frac{1}{k} = 0$$

By the ratio test, both series converge. Thus their sum converges as well.

(3) $\frac{\ln 2}{2^3} + \frac{\ln 3}{3^3} + \frac{\ln 4}{4^3} + \cdots$

**Solution:** $\frac{\ln 2}{2^3} + \frac{\ln 3}{3^3} + \frac{\ln 4}{4^3} + \cdots = \sum_{n=2}^{\infty} \frac{\ln n}{n^3}$. We will use the integral test.

$$\int_2^{\infty} \frac{\ln x}{x^3} dx = \lim_{b\to\infty} \int_2^b \frac{\ln x}{x^3} dx$$
$$= \lim_{b\to\infty} -\frac{\ln x}{2x^2}|_2^b + \frac{1}{2}\int_2^b \frac{1}{x^3} dx \text{(by integration by parts)}$$
$$= \lim_{b\to\infty} -\frac{\ln x}{2x^2}|_2^b - \frac{1}{4}\frac{1}{x^2}|_2^b$$
$$= \lim_{b\to\infty} -\frac{\ln b}{2b^2} + \frac{\ln 2}{8} - \frac{1}{4}(\frac{1}{b^2} - \frac{1}{2^2})$$
$$= \frac{\ln 2}{8} - \frac{1}{4}(-\frac{1}{2^2}) \text{(by L'Hopital's rule)}$$

Thus the series converges.

(4) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2} + \frac{1}{n}\right)^n$

**Solution:**

$$\lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{n}\right) = \frac{1}{2} < 1$$

Thus the series converges by the root test.

(5) $\sum_{n=2}^{\infty} \left( \frac{1}{\ln n} \right)^n$

**Solution:**

$$\lim_{n \to \infty} \frac{1}{\ln n} = 0 < 1$$

So, the series converges by the root test.