Worksheet #7

Determine if the series converges or diverges. State which test you use.

(1) $\sum_{k=1}^{\infty} \frac{k^2}{1+k^3}$

Solution: We will use the integral test.

$$\begin{aligned}
\int_1^{\infty} \frac{x^2}{1+x^3} dx &= \lim_{b\to\infty} \int_1^b \frac{x^2}{1+x^3} dx \\
&= \lim_{b\to\infty} \int_2^{1+b^3} \frac{-1}{3u} du \quad \text{where } u = 1+x^3 \\
&= \lim_{b\to\infty} \frac{-1}{3} \ln|u| |_2^{1+b^3} du \\
&= \lim_{b\to\infty} \frac{-1}{3} \left(\ln|1+b^3| - \ln(2)\right) \to -\infty
\end{aligned}$$

Thus the integral is divergent. This implies that the series is also divergent.

(2) $\sum_{k=1}^{\infty} \frac{k^2}{e^k}$

Solution: We will use the integral test.

$$\begin{aligned}
\int_1^{\infty} x^2 e^{-x} dx &= \lim_{b\to\infty} \int_1^b x^2 e^{-x} dx \\
&= \lim_{b\to\infty} \left(-x^2e^{-x} + 2\left(-xe^{-x} - e^{-x}\right)\right) |_1^b (\text{integrating by parts twice}) \\
&= \lim_{b\to\infty} \left[\left(-b^2e^{-b} + 2\left(-be^{-b} - e^{-b}\right)\right) - \left(-1^2e^{-1} + 2\left(-1e^{-1} - e^{-1}\right)\right)\right]
\end{aligned}$$

Now, $\lim_{b\to\infty} -b^2e^{-b} = 0$ by L'Hopital's rule twice. Also, $\lim_{b\to\infty} be^{-b} = 0$ by L'Hopital's rule. We know that $\lim_{b\to\infty} e^{-b} = 0$. Therefore, the integral converges. Thus the integral test tells us that the series converges.

(3) $\sum_{k=1}^{\infty} \frac{\sqrt{2n+1}}{n^3-4}$

Solution: We use the limit comparison test. Let $b_n = \frac{1}{n^{5/2}}$. Now

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\sqrt{2n+1}n^{5/2}}{n^3-4} \\
&= \lim_{n\to\infty} \sqrt{\frac{(2n+1)n^5}{(n^3-4)^2}} = \sqrt{2}
\end{aligned}$$

Therefore by limit comparison test, the series converges.

(4) $\displaystyle\sum_{k=5}^{\infty} \frac{1000}{k(\ln k)^2}$
Solution: We use the integral test. First, we will need a u-sub. $u = \ln x$

$$\begin{aligned}\int_5^{\infty} \frac{1000}{x(\ln x)^2} dx &= \lim_{b\to\infty} \int_{\ln 5}^{\ln b} \frac{1000}{u^2} du \\ &= \lim_{b\to\infty} -\frac{1000}{u}\Big|_{\ln 5}^{\ln b} \\ &= \lim_{b\to\infty} -\frac{1000}{\ln 5} + \frac{1000}{\ln b} = -\frac{1000}{\ln 5}\end{aligned}$$

Therefore, by the integral test, the series converges.

(5) $\displaystyle\sum_{n=1}^{\infty} \frac{n+3}{n^2\sqrt{n}}$
Solution:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n+3}{n^2\sqrt{n}} &= \sum_{n=1}^{\infty} \frac{n}{n^2\sqrt{n}} + \sum_{n=1}^{\infty} \frac{3}{n^2\sqrt{n}} \\ &= \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} + \sum_{n=1}^{\infty} \frac{3}{n^2\sqrt{n}}\end{aligned}$$

Both series converge by P-test.