Worksheet #6

(1) Use the monotonic convergence theorem to prove the sequences converge.

(a) $a_n = \frac{4n-3}{2^n}$

Solution: First, we need to determine if this is an increasing or decreasing sequence. We look to $a_{n+1} = \frac{4n+1}{2^{n+1}}$ and $a_n = \frac{4n-3}{2^n} = \frac{2(4n-3)}{2^{n+1}}$. For n large, $a_n > a_{n+1}$. Therefore, it is a decreasing sequence.

Next, we note that $a_n > 0$ for all n. Thus the sequence is bounded from below. Therefore by the monotonic convergence theorem, the sequence converges.

(b) $a_{n+1} = 1 + \frac{1}{2}a_n$ where $a_1 = 1$. (This is a recurrence relation. See page 723 for an example.)

Solution: To show that this sequence is increasing and that it is bounded we use mathematical induction. First lets look at a couple of terms in the sequence. $a_1 = 1$, $a_2 = \frac{3}{2}$, $a_3 = \frac{7}{4}$, and $a_4 = \frac{15}{8}$.

Now to prove it is an increasing sequence. We know $a_1 < a_2$. Lets assume that $a_{k-1} < a_k$. Now,

$$\begin{aligned}
\frac{1}{2}a_{k-1} &< \frac{1}{2}a_k \\
1 + \frac{1}{2}a_{k-1} &< 1 + \frac{1}{2}a_k \\
a_k &< a_{k+1}.
\end{aligned}$$

Thus the sequence is increasing.

Now to prove the sequence is bounded by 2. We know $a_1 < 2$. Lets assume $a_k < 2$. Now we must prove $a_{k+1} < 2$.

$$\begin{aligned}
a_k &< 2 \\
\frac{1}{2}a_k &< 1 \\
1 + \frac{1}{2}a_k &< 1 + 1 = 2 \\
a_{k+1} &< 2.
\end{aligned}$$

Thus the squence is bounded from above. Therefore by the monotonic convergence theorem, the sequence is convergent.

(2) Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

(a) $\int_2^3 \frac{1}{\sqrt{3-x}} dx$

Solution:

$$\begin{aligned}\int_2^3 \frac{1}{\sqrt{3-x}} dx &= \lim_{b \to 3^-} \int_2^b \frac{1}{\sqrt{3-x}} dx \\ &= \lim_{b \to 3^-} -2\sqrt{3-x}\Big|_2^b \\ &= \lim_{b \to 3^-} -2\left(\sqrt{3-b} - 1\right) \\ &= 2\end{aligned}$$

Thus the integral is convergent.

(b) $\int_{-\infty}^{\infty} \cos(\pi t) dt$

Solution:

$$\begin{aligned}\int_{-\infty}^{-\infty} \cos(\pi t) dt &= \lim_{b \to -\infty} \int_b^0 \cos(\pi t) dt + \lim_{c \to \infty} \int_0^c \cos(\pi t) dt \\ &= \lim_{b \to -\infty} \frac{1}{\pi} \sin(\pi t)|_b^0 + \lim_{c \to \infty} \frac{1}{\pi} \sin(\pi t)|_0^c \\ &= \lim_{b \to -\infty} \frac{1}{\pi} \sin(\pi b) + \lim_{c \to \infty} \frac{1}{\pi} \sin(\pi c)\end{aligned}$$

These limits do not exist thus the integral is not convergent.

(c) $\int_1^{\infty} \frac{\ln x}{x} dx$

Solution: First we rewrite the integral with a limit.

$$\int_1^{\infty} \frac{\ln x}{x} dx = \lim_{b \to \infty} \int_1^b \frac{\ln x}{x} dx$$

Now we must do integration by parts to evaluate the integral.
Let $u = \ln x$ and $dv = \frac{1}{x} dx$. Then $du = \frac{1}{x} dx$ and $v = \ln x$. Now,

$$\int_1^b \frac{\ln x}{x} dx = (\ln x)^2|_1^b - \int_1^b \frac{\ln x}{x} dx.$$

Combining like terms, we find $\int_1^b \frac{\ln x}{x} dx = \frac{1}{2}(\ln x)^2|_1^b = \frac{1}{2}(\ln b)^2$. Therefore,

$$\int_1^{\infty} \frac{\ln x}{x} dx = \lim_{b \to \infty} \frac{1}{2}(\ln b)^2$$

diverges.

(d) $\displaystyle\int_4^\infty \frac{1}{\sqrt{x-3}}dx$
Solution:

$$\begin{aligned}\int_4^\infty \frac{1}{\sqrt{x-3}}dx &= \lim_{b\to\infty}\int_4^b \frac{1}{\sqrt{x-3}}dx \\ &= \lim_{b\to\infty} 2\sqrt{x-3}|_4^b \\ &= \lim_{b\to\infty} 2(\sqrt{b-3}-1)\end{aligned}$$

The limit goes infinty. Therefore the integral is divergent.