Worksheet #26

(1) Find all critical points. Indicate whether each point gives a local minimum, local maximum, or a saddle point.

$$f(x,y) = xy^2 - 6x^2 - 3y^2$$

Solution: First, we must find the critical points (ie. where does $f_x = 0$ and $f_y = 0$.)

$$f_x = y^2 - 12x \quad f_y = 2xy - 6y$$

Setting each of these equal to zero and solving for x and y, we find the critical points are $(0,0)$, $(3,6)$, and $(3,-6)$.

Now we use the second derivative test to classify the points.

Note

$$f_{xx} = -12 \quad f_{xy} = 2y \quad f_{yy} = 2x - 6$$

Case 1: $(0,0)$ $D(0,0) = 72 > 0$ Since $f_{xx} < 0$, $(0,0)$ is a local max.
Case 2: $(3,6)$ $D(3,6) = -12^2 < 0$ This implies $(3,6)$ is a saddle point.
Case 3: $(3,-6)$ $D(3,-6) = -12^2 < 0$ This implies $(3,-6)$ is a saddle point.

(2) Find the global minimum value and global maximum value of $f(x,y) = 4x + 6y - x^2 - y^2$ on $S = \{(x,y) : 0 \leq x \leq 4, 0 \leq y \leq 5\}$ and indicate where they occur.

Solution: First, we find the critical points (ie. where does $f_x = 0$ and $f_y = 0$.)

$$f_x = 4 - 2x \quad f_y = 6 - 2y$$

Setting each of these equal to zero and solving for x and y, we find the critical point is $(2,3)$. This point is in S so we use the second derivative test to determine if it is a local max or min.

$$f_{xx} = -2 \quad f_{xy} = 0 \quad f_{yy} = -2$$

Since $D(2,3) = 4 > 0$ and $f_{xx} < 0$, $(2,3)$ is a local max. $f(2,3) = 13$.

Now we must check the boundary.

$x = 0$ *edge*

$$g_1(y) = f(0,y) = 6y - y^2$$

$y = 3$ is a critical point. $g_1'' < 0$ implies $y = 3$ is a local maximum. $f(0,3) = 9$.

$x = 4$ *edge*

$$g_2(y) = f(4,y) = 6y - y^2$$

$y = 3$ is a critical point. $g_2'' < 0$ implies $y = 3$ is a local maximum. $f(4,3) = 9$.

$y = 0$ *edge*

$$g_3(x) = f(x,0) = 4x - x^2$$

$x = 2$ is a critical point. $g_3'' < 0$ implies $x = 2$ is a local maximum. $f(2,0) = 4$.

$y = 5$ *edge*

$$g_4(x) = f(x,5) = 4x - x^2 + 5$$

$x = 2$ is a critical point. $g_4'' < 0$ implies $x = 2$ is a local maximum. $f(2,5) = 9$.

Checking the corners, we find $f(0,0) = f(4,0) = 0$.

Thus, the global max value is 13 at $(2,3)$ and the global min value is 0 at $(0,0)$ and $(4,0)$.

(3) Find the 3-dimensional vector with length 9, the sum of whose components is a maximum.

Solution: Let $< x, y, z >$ denote the vector we are searching for. We know that

$$L^2 = x^2 + y^2 + z^2 = 81$$

and we want to maximize

$$S(x, y, z) = x + y + z.$$

Solving for z in L^2, we find $z = \pm\sqrt{81 - x^2 - y^2}$. We want S to be maximized thus z must be positive, ie. $z = \sqrt{81 - x^2 - y^2}$. Plugging this into S, we find $S(x, y) = x + y + \sqrt{81 - x^2 - y^2}$ for $0 \leq x^2 + y^2 \leq 9$.

We must find the critical points.

$$S_x = 1 - \frac{x}{\sqrt{81 - x^2 - y^2}} \quad S_y = 1 - \frac{y}{\sqrt{81 - x^2 - y^2}}$$

Setting both equal to zero, we find

$$x = \sqrt{81 - x^2 - y^2} \quad y = \sqrt{81 - x^2 - y^2}.$$

Thus the only critical point is $x = y = 3\sqrt{3}$. We find $z = 3\sqrt{3}$ and

$$S(3\sqrt{3}, 3\sqrt{3}, 3\sqrt{3}) = 9\sqrt{3}.$$

Checking the boundary ($x = 3\cos t, y = 3\sin t$), we find the max S can be is $\frac{18}{\sqrt{2}} < 9\sqrt{3}$.

Thus the vector we are looking for is $3\sqrt{3} < 1, 1, 1 >$.