Worksheet #25

(1) Find the equation of the tangent plane and the normal line to the surface $x+y+z=e^{xyz}$ at $(0,0,1)$.
Solution: The normal vector is the gradient evaluated at the point. Let $F(x,y,z) = x+y+z-e^{xyz}$. The gradient is

$$\nabla F(x,y,z) =< F_x, F_y, F_z >=< 1-yze^{xyz}, 1-xze^{xyz}, 1-xye^{xyz} >$$

$$\nabla F(0,0,1) =< 1,1,1 >$$

The tangent plane is

$$< 1,1,1 > \cdot < x, y, z-1 >= 0$$

or $x+y+z-1=0$. The normal line (i.e. the line with direction given by the normal vector through $(0,0,1)$ is

$$\mathbf{r}(t) =< t, t, 1+t > .$$

(2) Find the directional derivative of $f(x,y) = e^x \sin y$ at $P(0,\pi/4)$ in the direction of $\mathbf{a} =< 1, \sqrt{3} >$.
Solution: We know $D_{\mathbf{u}} f = \nabla f|_P \cdot \mathbf{u}$.

$$\nabla f(x,y) =< e^x \sin y, e^x \cos y > \quad \nabla f(0,\pi/4) =< \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} >$$

$$\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|} =< 1/2, \frac{\sqrt{3}}{2} >$$

$$D_{\mathbf{u}} f = \nabla f|_P \cdot \mathbf{u} =< \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} > \cdot < 1/2, \frac{\sqrt{3}}{2} >= \frac{\sqrt{2}+\sqrt{6}}{4}$$

(3) Find a unit vector in the direction in which $f(x,y,z) = x^2yz$ increases most rapidly at $P(1,-1,2)$.
Solution:

$$\nabla f(x,y,z) =< 2xyz, x^2z, x^2y > \quad \nabla f|_P =< -4, 2, -1 >$$

$$|\nabla f|_P| = \sqrt{16+4+1} = \sqrt{21}$$

The unit vector in the direction of max increase is $\mathbf{u} = \frac{1}{\sqrt{21}} < -4, 2, -1 >$

(4) The elevation of a mountain above sea level at (x,y) is $3000e^{-(x^2+2y^2)/100}$ meters. The positive x-axis points east and the positive y-axis points north. A climber is directly above $(10,10)$. If the climber moves northwest, will she ascend or descend and at what slope?
Solution: Let $f(x,y) = 3000e^{-(x^2+2y^2)/100}$. We must look at the gradient of f to determine whether northwest is the ascend or descend.

$$\nabla f =< -60xe^{-(x^2+2y^2)/100}, -120ye^{-(x^2+2y^2)/100} >$$

$$\nabla f|_{(10,10)} = -600e^{-3} < 1, 2 >$$

$$\mathbf{u} = \frac{1}{\sqrt{2}} < -1, 1 >$$

The vector **u** points in the northwest direction which means the climber is ascending. The slope is given by

$$D_{\mathbf{u}}f = \frac{-600e^{-3}}{\sqrt{2}} < 1, 2 > \cdot < -1, 1 >= \frac{-600e^{-3}}{\sqrt{2}}(-1+2) = \frac{-600e^{-3}}{\sqrt{2}}.$$