Worksheet #20

(1) Find the length of the curve

$$\mathbf{r}(t) = \left\langle 2t, t^2, \frac{1}{3}t^3 \right\rangle$$

for $0 \leq t \leq 1$.

Solution: We need to evaluate $L = \int_0^1 |\mathbf{r}'(t)|dt$. First,

$$\mathbf{r}'(t) = \langle 2, 2t, t^2 \rangle .$$

Hence,

$$|\mathbf{r}'(t)| = \sqrt{4 + 4t^2 + t^4} = \sqrt{(t^2+2)^2} = t^2 + 2$$

Thus,

$$L = \int_0^1 |\mathbf{r}'(t)|dt = \int_0^1 (t^2+2)dt = \frac{t^3}{3} + 2t|_0^1 = 2\frac{1}{3}$$

(2) Find the length of the curve intersection of the cylinder $4x^2 + y^2 = 4$ and the plane $x + y + z = 2$.

Solution: First, we need a parametric equation of the cylinder. To get this, we rewrite the equation of the cylinder as $x^2 + \left(\frac{y}{2}\right)^2 = 1$. From this equation it is easy to see $x = \cos t$ and $y = 2\sin t$ for $0 \leq t \leq 2\pi$. Plugging these values into the equation of the plane, we find $z = 2 - \cos t - 2\sin t$. Thus the curve is given by the vector function

$$\mathbf{r}(t) =< \cos t, 2\sin t, 2 - \cos t - 2\sin t > .$$

Hence,

$$\mathbf{r}'(t) =< -\sin t, 2\cos t, \sin t - 2\cos t > .$$

We know the length of this curve is given by

$$\begin{aligned} L &= \int_0^{2\pi} |\mathbf{r}'(t)|dt \\ &= \int_0^{2\pi} \sqrt{\sin^2 t + 4\cos^2 t + (\sin t - 2\cos t)^2 dt} \\ &= \int_0^{2\pi} \sqrt{2(\sin^2 t + 4\cos^2 t - 2\cos t \sin t)dt} \\ &= \int_0^{2\pi} \sqrt{2(4 - 2\cos t \sin t)}dt \end{aligned}$$

(3) Find the unit tangent vector $\mathbf{T}(t)$ and the curvature for the curve

$$\mathbf{r}(t) =< t^2, \sin t - t\cos t, \cos t + t\sin t >, \quad t > 0.$$

Solution:

$$\mathbf{r}'(t) =< 2t, \cos t - (-t\sin t + \cos t), -\sin t + (t\cos t + \sin t) >=< 2t, t\sin t, t\cos t >$$

$$\begin{aligned} |\mathbf{r}'(t)| &= \sqrt{(2t)^2 + (t\sin t)^2 + (t\cos t)^2} \\ &= \sqrt{4t^2 + t^2(\sin^2 t + \cos^2 t)} = \sqrt{5}t \end{aligned}$$

With this information, we find the unit tangent vector

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{5}} < 2, \sin t, \cos t > .$$

To compute the curvature, we need $|\mathbf{T}'(t)|$.

$$\mathbf{T}'(t) = \frac{1}{\sqrt{5}} < 0, \cos t, -\sin t >$$

So

$$|\mathbf{T}'(t)| = \frac{1}{\sqrt{5}}$$

Thus the curvature κ is

$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1}{5t}.$$