Worksheet #18

(1) Find a parametric equation for the line through $(1, -2, 3)$ and $(4, 5, 6)$.
Solution: $\mathbf{v} =< 4-1, 5+2, 6-3 >=< 3, 7, 3 >$. So the line is given by $x = 1 + 3t$, $y = -2 + 7t$ and $z = 3 + 3t$.

(2) Write both the parametric equations and the symmetric equations for the line through the point $(1, 1, 1)$ parallel to the vector $< -10, -100, -1000 >$.
Solution: The parametric equation of the line is

$$(x, y, z) = (1 - 10t, 1 - 100t, 1 - 1000t).$$

The symmetric equation of the line is

$$\frac{x-1}{-10} = \frac{y-1}{-100} = \frac{z-1}{-1000}.$$

(3) Show that the lines

$$\frac{x-1}{-4} = \frac{y-2}{3} = \frac{z-4}{-2}$$

and

$$\frac{x-2}{-1} = \frac{y-1}{1} = \frac{z+2}{6}$$

intersect and find the equation of the plane they determine.
Solution: The parametric equation of the first line is

$$\begin{aligned} x &= 1 - 4t \\ y &= 2 + 3t \\ z &= 4 - 2t \end{aligned}$$

The parametric equation of the second line is

$$\begin{aligned} x &= 2 - s \\ y &= 1 + s \\ z &= -2 + 6s \end{aligned}$$

Setting the x and y equations equal to each other, we get two equations with two unknowns.

$$\begin{aligned} 1 - 4t &= 2 - s \\ 2 + 3t &= 1 + s \end{aligned}$$

Solving we find $t = 0$ and $s = 1$. Pluging $t = 0$ in to the first line, we find the intersect is $(x, y, z) = (1, 2, 4)$.

The direction vector for the first line is $\mathbf{v}_1 =< -4, 3, -2 >$. The direction vector for the second line is $\mathbf{v}_2 =< -1, 1, 6 >$. We need to find a vector normal to the plane determined by the two lines. It is

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 3 & -2 \\ -1 & 1 & 6 \end{bmatrix} = 20\mathbf{i} + 26\mathbf{j} - \mathbf{k}$$

The plane determined by the two lines is

$$\mathbf{n} \cdot < x-1, y-2, z-4 >= 20(x-1) + 26(y-2) - (z-4) = 0.$$

(4) Let $3x - 2y + z = 1$ and $2x + y - 3z = 3$ be two planes. Find the parametric equation for the line of intersection of the planes. Also find the angle between the two planes.
Solution: First, we need to determine a point on the line of intersection. We choose to find the] point where both lines intersect the xy-plane. Setting $z = 0$ and solving for x and y, we find the point $(1, 1, 0)$.

Next, we need to determine the direction. For the first plane, $\mathbf{n}_1 =< 3, -2, 1 >$. For the second plane, $\mathbf{n}_2 =< 2, 1, -3 >$. The direction is given by

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix} =< 5, 11, 7 >$$

Thus the line is given by

$$(x, y, z) = (1 + 5t, 1 + 11t, 7t)$$

for $-\infty < t < \infty$.

To find the angle between the two planes, we know

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{1}{14}.$$

Solving for θ, we find the angle between the two planes to be $\arccos(1/14)$.