Worksheet #12

(1) Find the Mclaurin series for $f(x) = e^{-2x}$.

Solution: We know the Mclaurin series for e^y is $e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$. Thus

$$e^{-2x} = \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-2)^n x^n}{n!}.$$

(2) Find the Mclaurin series for $f(x) = \sqrt{1-x}$.
Solution: We will use the binomial series formula. Thus the series is

$$\sqrt{1-x} = \sum_{k=0}^{\infty} \binom{1/2}{k} (-1)^k x^k.$$

(3) Find the 4^{th} degree Taylor polynomial of $f(x) = \frac{1}{x}$ centered at $a = -3$. Give a bound for the error.
Solution:

$$\begin{aligned} f(x) &= \frac{1}{x} & f(-3) &= -\frac{1}{3} \\ f'(x) &= -\frac{1}{x^2} & f'(-3) &= -\frac{1}{3^2} \\ f''(x) &= 2\frac{1}{x^3} & f''(-3) &= -2\frac{1}{3^3} \\ f^{(3)}(x) &= -6\frac{1}{x^4} & f^{(3)}(-3) &= -6\frac{1}{3^4} \\ f^{(4)}(x) &= 24\frac{1}{x^5} & f^{(4)}(-3) &= -24\frac{1}{3^5} \end{aligned}$$

Thus

$$T_4(x) = -\frac{1}{3} - \frac{1}{3^2}(x+3) - \frac{2}{3^3}(x+3)^2 - \frac{6}{3^4}(x+3)^3 - \frac{24}{3^5}(x+3)^4$$

The error bound $R_4(x)$ is such that $|R_4(x)| \leq \frac{M}{5!}|x+3|^5$ where $M = max_{-d-3 \leq x \leq d-3} |\frac{24*5}{x^6}| = \frac{24*5}{(3-d)^6}$.

(4) Find the Taylor series of $f(x) = x - x^3$ centered at $a = 2$ and the radius of convergence.
Solution:

$$\begin{aligned} f(x) &= x - x^3 & f(2) &= -6 \\ f'(x) &= 1 - 3x^2 & f'(2) &= -11 \\ f''(x) &= -6x & f''(2) &= -12 \\ f^{(3)}(x) &= -6 & f^{(3)}(2) &= -6 \\ f^{(n)}(x) &= 0 & f^{(n)}(x) &= 0 \text{ for } n > 3 \end{aligned}$$

Thus $f(x) = -6 - 11(x-2) - \frac{12}{2!}(x-2)^2 - \frac{6}{3!}(x-2)^3$ for all x.