Worksheet #11

Find the power series representation for $f(x)$ and specify the radius of convergence.

(1) $\arctan x = \int \frac{1}{1+x^2} dx$

Solution: We know that $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$. Thus $\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$. Integrating term by term, we find $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$.

(2) $f(x) = \frac{x^2}{1-x^4}$

Solution: We know that $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$. Thus

$$f(x) = x^2 \left(\frac{1}{1-x^4} \right) = x^2 \sum_{n=0}^{\infty} (x^4)^n = \sum_{n=0}^{\infty} x^{4n+2}$$

(3) $f(x) = \ln(x^2+4)$

Solution:

We know

$$\ln(1+y) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$$

Now

$$\begin{aligned} f(x) &= \ln(4(1+\frac{x^2}{4})) \\ &= \ln(4) + \ln(1+\frac{x^2}{4}) \\ &= \ln(4) + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{n4^n} \end{aligned}$$

To determine the radius of convergence, we need $\left| \frac{x^2}{4} \right| < 1$.

This means $R = 2$.

(4) $f(x) = \dfrac{1+x}{(1-x)^2}$

Solution:

$$\begin{aligned}
\frac{1+x}{(1-x)^2} &= (1+x)\frac{d}{dx}\left(\frac{1}{1-x}\right) \\
&= (1+x)\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) \\
&= (1+x)\left(\sum_{n=0}^{\infty} nx^{n-1}\right) \\
&= (1+x)\left(\sum_{n=1}^{\infty} nx^{n-1}\right) \\
&= \sum_{n=1}^{\infty} n(x^{n-1}+x^n)
\end{aligned}$$

The radius of convergence is $R = 1$.