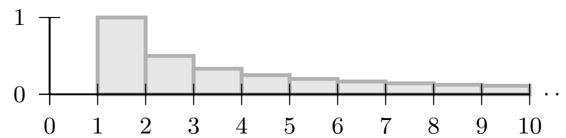
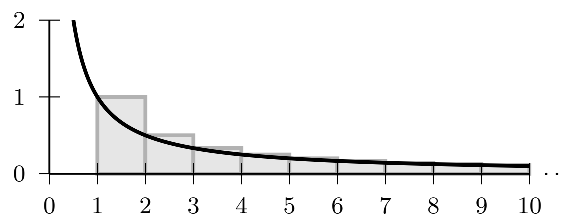
2.4. IMPROPER INTEGRALS AND THE INTEGRAL TEST

In this section we discuss a very simple, but powerful, idea: in order to prove that certain series converge or diverge, we may compare them to integrals. There are a few important caveats with this comparison, which we will make note of when we present the Integral Test formally. To motivate this test, we return to the harmonic series $\sum 1/n$. In Section 2.2, we saw Nicolas Oresme's classic proof that the harmonic series diverges. Here, we present another proof, which will generalize to handle many more series.

To begin, let us represent the series $\sum 1/n$ as the total area contained in an infinite sequence of $1 \times 1/n$ rectangles. Beginning with the first rectangle stretching from $x = 1$ to $x = 2$ and placing the rectangles next to each other, we get the following.



We now approximate the area under these rectangles. In this case, we only have to observe that the function $1/x$ lies below the tops of these rectangles for $x \geqslant 1$, as shown below.



Therefore, there is more area under the rectangles than under the function $1/x$. As we know that area under a curve is given by an integral, to find the area under $1/x$ for $x \geqslant 1$, we need to evaluate

$$\int_1^\infty \frac{1}{x}\,dx.$$

This type of an integral may be unfamiliar because it involves infinity, and for this reason, integrals of this type are called *improper integrals*[†]. Since we can't simply take the anti-derivative of $1/x$ and plug in ∞, we do the next best thing. We *define* the improper integral as a limit:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx.$$

[†]These are not the only type of improper integrals. Others involve integrating near a vertical asymptote of a function. See Exercises 36–43 for more examples of improper integrals.

Using this definition, we have another argument for why $\sum 1/n$ diverges:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n} &\geqslant \int_1^{\infty}\frac{1}{x}\,dx\\ &= \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx\\ &= \lim_{b\to\infty}\ \ln x\big|_1^b\\ &= \lim_{b\to\infty}\ \ln b-\ln 1\\ &= \infty.\end{aligned}$$

In general, if we have a function $f(x)$ defined from $x=a$ to $x=\infty$, we define

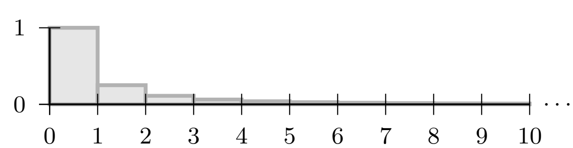
$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx,$$

and we say that this improper integral *converges* if the limit converges, and that it *diverges* if the limit diverges. The reader should note the similarity between this definition and the definition we made for the convergence of a series.

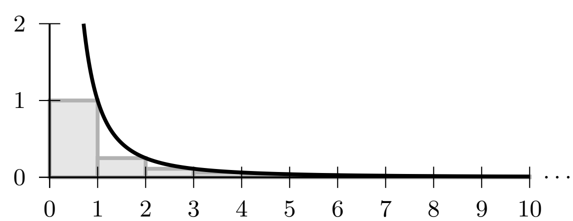
Comparing series to integrals can also be used to show that they converge, as we illustrate in the next example.

Example 1. Show that $\sum 1/n^2$ converges by comparing it to an integral.

Solution. We do roughly the same thing as we did with the harmonic series, but here, since we are to show that the series converges, we want the area in our rectangles to be *less* than the area under the curve. For this reason, we begin by placing $1 \times 1/n^2$ rectangles starting at $x=0$:



Now we have exactly what we want, because the area under these rectangles is strictly less than the area under $1/x^2$ for $x \geqslant 0$:



This suggests that we should integrate ${}^{1}\!/_{x^2}$ from $x = 0$ to $x = \infty$ to get an upper bound on $\sum {}^{1}\!/_{n^2}$. However, since ${}^{1}\!/_{x^2}$ has a vertical asymptote at $x = 0$, such an integral would be doubly improper and actually diverges, as shown by Exercises 37 and 38, so can't be used to bound $\sum {}^{1}\!/_{n^2}$ from above. To get around this problem, we can simply pull off the first term of the series, and compare the rest to the integral of ${}^{1}\!/_{x^2}$ from $x = 1$ to $x = \infty$:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2} &= 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \\ &\leqslant 1 + \int_1^{\infty} \frac{1}{x^2}\,dx \\ &= 1 + \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2}\,dx \\ &= 1 + \lim_{b\to\infty} -\frac{1}{x}\bigg|_1^b \\ &= 2 - \lim_{b\to\infty} \frac{1}{b} \\ &= 2.\end{aligned}$$

It is tempting to conclude right now that $\sum {}^{1}\!/_{n^2}$ converges, because we know that it is at most 2, but this would be reckless. Remember in Example 6 of Section 2.2 we presented a series — the series $\sum(-1)^{n+1}$ — which has bounded partial sums but does not converge.

However, in this case, ${}^{1}\!/_{n^2}$ is positive for all n, so the partial sums $\{s_n\}$ of $\sum {}^{1}\!/_{n^2}$ are monotonically increasing. By the above argument, these partial sums are bounded: s_n lies between 0 and 2 for all n, and therefore we know by the Monotone Convergence Theorem that the sequence $\{s_n\}$ converges to a limit, and thus $\sum {}^{1}\!/_{n^2}$ converges. ●

While we have shown that $\sum {}^{1}\!/_{n^2}$ converges, we have not computed its value. For series that aren't geometric, such questions are generally extremely difficult, and $\sum {}^{1}\!/_{n^2}$ is no exception. Finding $\sum {}^{1}\!/_{n^2}$ became known as the Basel problem after it was posed by Pietro Mengoli (1626–1686) in 1644. In 1735, at the age of twenty-eight, Leohnard Euler showed that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

one of the first major results in what would be a marvelous career. We discuss his first proof (which has been incredibly influential despite the fact that it contains a flaw) in Exercises 54–58 of Section 3.3. Euler began his exploration of the Basel problem by computing the sum to 17 decimal places (which in itself was quite a feat, accomplished by viewing the series as an integral), a bit like we did in Section 2.2 to guess the sum of the alternating harmonic series. Amazingly, without any aid like the *Inverse Symbolic Calculator*, Euler recognized that this approximation looked like ${}^{\pi^2}\!/_{6}$! This gave Euler a significant advantage in finding the solution, since he knew what the answer should be.

Euler went on to find formulas for $\sum 1/n^p$ for all even integers p. But what about the odd integers? For a *very* long time, mathematicians could not even prove that $\sum 1/n^3$ was irrational, let alone express it in terms of well-known constants. In 1978, Roger Apéry (1916–1994) announced that he had a proof of this result. However, Apéry was not well-known and there were significant doubts that his proof could be correct. Apéry fed this suspicion by giving a strange talk announcing his proof, one of the key ingredients of which was the equation

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n!)^2}{n^3(2n)!}.$$

When asked how he derived this equation, Apéry is alleged to have replied "they grow in my garden." Nevertheless, he completed his proof, stunning the mathematical establishment. The analogous question about the irrationality of $\sum 1/n^p$ for odd integers $p \geqslant 5$ remains unsolved.

It is now time to generalize our two examples and make a test out of them. First we must decide what was special about $\sum 1/n$ and $\sum 1/n^2$ that allowed us to make the comparisons we made. In the case of $\sum 1/n$, we needed that the function $1/x$ lies below the rectangles we formed. This relies on the fact that $1/x$ is decreasing. Similarly, in the case of $\sum 1/n^2$ we needed that the function $1/x^2$ lies above the rectangles we formed. Because these rectangles were slid over by one unit, this too relies on the fact that $1/x^2$ is decreasing. We also used, in the $\sum 1/n^2$ case, the fact that $1/x^2$ is positive. Finally, we need to be able to evaluate the integrals. We could add this as a hypothesis, but in the interest of simplicity, we simply require that our functions be continuous, which guarantees that they can be integrated. Under these conditions, we have the following test.

The Integral Test. Suppose that f is a positive, decreasing, and continuous function, and that $a_n = f(n)$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ converges.

Proof. As in our two examples, we can sandwich the partial sums s_n between two improper integrals:

$$\int_1^n f(x)\,dx \leqslant s_n \leqslant a_1 + \int_1^{n-1} f(x)\,dx.$$

Now since we are proving an "if and only if" statement, we have two things to prove. First, suppose that

$$\int_1^{\infty} f(x)\,dx$$

converges. Then, by using the upper-bound, we have

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n \leqslant \lim_{n\to\infty} a_1 + \int_1^{n-1} f(x)\,dx < \infty.$$

Because $a_n = f(n)$ is positive, we know that the partial sums s_n are monotonically increasing, so since the above inequality shows that the sequence $\{s_n\}$ is bounded, the Monotone Convergence Theorem implies that $\{s_n\}$ has a limit. This proves that $\sum a_n$ converges if the improper integral $\int_1^{\infty} f(x)\,dx$ converges.

Now suppose that

$$\int_1^{\infty} f(x)\,dx$$

diverges. Using the lower-bound, we have

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n \geqslant \lim_{n\to\infty} \int_1^{n} f(x)\,dx = \infty.$$

Therefore, the sequence $\{s_n\}$ of partial sums diverges to ∞, so the series $\sum a_n$ diverges. □

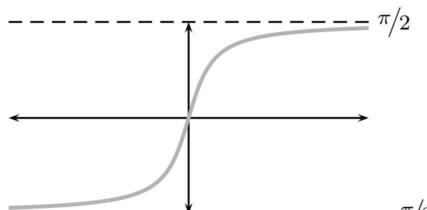
The Integral Test is a very powerful tool, but it has a serious drawback: we must be able to evaluate the improper integrals it requires. For example, how could we use the Integral Test to determine whether $\sum 4^n/n!$ converges? Nevertheless, there are numerous examples of series to which it applies.

Example 2. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converge or diverge?

Solution. We first evaluate the improper integral in the Integral Test:

$$\int_1^{\infty} \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_1^{b} \frac{dx}{x^2+1} = \lim_{b\to\infty} \arctan b - \arctan 1.$$

To evaluate this limit, it may be helpful to recall the plot of $\arctan$:



Therefore, we have

$$\lim_{b\to\infty} \arctan b - \arctan 1 = \pi/2 - \pi/4,$$

so the series $\sum 1/(n^2+1)$ converges by the Integral Test. ●

We began the section by considering $\sum 1/n$ and $\sum 1/n^2$. What about $\sum 1/n^p$ for other values of p? We can evaluate the integral of $1/x^p$, so the Integral Test can be used to determine which of these series converge. Because series of this form occur so often, we record this fact as its own test.

The p-Series Test. The series $\sum 1/n^p$ converges if and only if $p > 1$.

Proof. When $p = 1$, we already know that the series diverges ($\sum 1/n$ is the example we began the section with). For other values of p, we simply integrate the improper integral from the Integral Test:

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{b\to\infty}\int_1^b x^{-p}\,dx \\ &= \lim_{b\to\infty} \frac{1}{-p+1}x^{-p+1}\Big|_1^b \\ &= \left(\lim_{b\to\infty}\frac{1}{-p+1}b^{-p+1}\right) - \frac{1}{-p+1}\end{aligned}$$

If $p < 1$ then the function x^{-p+1} decreases to 0 (the exponent is negative), so in this case the limit above converges to $-1/{-p+1} = 1/{1-p}$. Therefore, by the Integral Test, $\sum 1/n^p$ converges if $p < 1$. On the other hand, if $p > 1$ then the function x^{-p+1} increases without bound, so in this case the limit above diverges to ∞, and so $\sum 1/n^p$ diverges if $p > 1$. □

We conclude this section with error estimates. Since improper integrals can be used to bound series, they can also be used to bound the tails of series, i.e., the error in a partial sum:

The Integral Test Remainder Estimates. Suppose that f is a positive, decreasing, and continuous function, and that $a_n = f(n)$. Then the error in the nth partial sum of $\sum a_n$ is bounded by an improper integral:

$$\left| s_n - \sum_{n=1}^{\infty} a_n \right| \leqslant \int_n^\infty f(x)\,dx.$$

The proof of the Integral Test Remainder Estimate is almost identical to the proof of the Integral Test itself, so we content ourselves with an example.

Example 3. Bound the error in using the fourth partial sum s_4 to approximate $\sum_{n=1}^{\infty} 1/n^2$.

Solution. The error in this case is the difference between s_n and the true value of the series:

$$\text{Error} = \left| s_n - \sum_{n=1}^{\infty} \frac{1}{n^2} \right|$$

By the remainder estimates, we have:

$$\begin{aligned} \text{Error} &\leqslant \int_4^{\infty} \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} \int_4^{b} \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} -\frac{1}{x}\bigg|_4^b \\ &= \frac{1}{4}. \end{aligned}$$

This is *not* a very good bound. As we mentioned earlier, Euler approximated the value of this series to within 17 decimal places. How many terms would we need to take to get the upper bound on the error from the Integral Test Remainder Estimates under 10^{-17}? ●

EXERCISES FOR SECTION 2.4

Arrange the quantities in Exercises 1–4 in order from least to greatest.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\quad 1 + \displaystyle\int_1^{\infty} \frac{dx}{x^2}$ $\quad \displaystyle\int_2^{\infty} \frac{dx}{x^2}$

2. $\displaystyle\sum_{n=9}^{\infty} \frac{1}{n^2+1}$ $\quad \displaystyle\int_{10}^{\infty} \frac{dx}{x^2+1}$ $\quad \displaystyle\int_9^{\infty} \frac{dx}{x^2+1}$

3. $\displaystyle\sum_{n=1}^{10} \frac{1}{\sqrt{n^7+2}}$ $\quad \displaystyle\int_1^{11} \frac{dx}{\sqrt{x^7+2}}$ $\quad \displaystyle\int_0^{10} \frac{dx}{\sqrt{x^7+2}}$

4. $\displaystyle\sum_{n=1}^{10} -n$ $\quad \displaystyle\int_0^{10} -x\,dx$ $\quad \displaystyle\int_1^{11} -x\,dx$

For Exercises 5–20, use the Integral Test to determine if the series converge or diverge, *or* indicate why the Integral Test cannot be used.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n+3}}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{4n}{(2n^2+3)^2}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{(2+5n)}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{n-\sqrt{n}}{n}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2-\sin n}}$

13. $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^2+1}$

14. $\sum_{n=2}^{\infty} \frac{\ln n}{n}$

15. $\sum_{n=1}^{\infty} \frac{2n+3}{\sqrt{n}}$

16. $\sum_{n=1}^{\infty} \frac{5}{n\sqrt{n}}$

17. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

18. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

19. $\sum_{n=2}^{\infty} \frac{1}{n \ln n(\ln \ln n)}$

20. $\sum_{n=2}^{\infty} \frac{1}{n \ln n(\ln \ln n)^2}$

By the Integral Test Remainder Estimates, how many terms would you need to use to approximate the sums in Exercises 21–24 to within $1/100$?

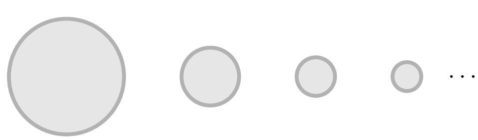
21. $\sum_{n=1}^{\infty} \frac{1}{n^4}$

22. $\sum_{n=1}^{\infty} \frac{1}{n^5}$

23. $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^2}$

24. $\sum_{n=1}^{\infty} \frac{5}{n\sqrt{n}}$

Exercises 25 and 26 concern an infinite sequence of circles which do not overlap and have radii 1, $1/2$, $1/3$, . . . , as shown below.



25. Is the total area inside all of the circles finite? (Note that you are not asked to find this total.)

26. Is the total circumference inside all of the circles finite? (Note that you are not asked to find this total.)

Exercises 27–32 require integration by parts. Use the Integral Test to determine if the series converge or diverge.

27. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

28. $\sum_{n=1}^{\infty} \frac{\ln n}{n^{7/6}}$

29. $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$

30. $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

31. $\sum_{n=1}^{\infty} \frac{1}{ne^{1/n}}$

32. $\sum_{n=1}^{\infty} \frac{n}{2^n}$

33. Use the Integral Test to verify that the geometric series $\sum_{n=1}^{\infty} ar^n$ converges for $0 < r < 1$.

34. In the Integral Test, we began both the series and the integral at 1 (technically, $n = 1$ and $x = 1$, respectively). Show that this is not necessary by proving the following. Suppose that f is a positive, decreasing, and continuous function, and that $a_n = f(n)$. Then

$$\sum_{n=1}^{\infty} a_n$$

converges if and only if there is some N so that the improper integral

$$\int_N^{\infty} f(x)\,dx$$

converges.

35. Use the fact that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n}\right) \end{aligned}$$

to prove that the alternating harmonic series converges using the Integral Test.

We have focused on only one type of improper integrals, which are called improper because their domains are infinite. However, there is another type, which are called improper because their integrands have vertical asymptotes. To begin with, suppose that $f(x)$ is continuous on the interval $(a, b]$ but discontinuous at $x = a$. Then we define the integral of $f(x)$ from a to b as a limit:

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

Use this definition to evaluate the integrals in Exercises 36–39.

36. $\displaystyle\int_0^1 \sqrt{x}\,dx$

37. $\displaystyle\int_0^1 x^2\,dx$

38. $\displaystyle\int_0^\infty x^2\,dx$

39. $\displaystyle\int_0^1 x^{1.001}\,dx$

If instead $f(x)$ is continuous on the interval $[a, b)$ but discontinuous at $x = b$, then we consider the limit as the upper-bound of the integral approaches b from below:

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

Use this definition to evaluate the integrals in Exercises 40–41.

40. $\displaystyle\int_0^1 \frac{1}{1-x}\,dx$

41. $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

Finally, it could be the case that $f(x)$ has a vertical asymptote between the bounds a and b, say at $x = c$ for $a < c < b$. In this case, assuming that $f(x)$ is continuous on both $[a, c)$ and $(c, b]$, we break the integral in two:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

We then evaluate each of these improper integrals using the rules above. Using this definition, evaluate the integrals in Exercises 42–43.

42. $\displaystyle\int_0^2 \frac{1}{(1-x)^2}\,dx$

43. $\displaystyle\int_{-1}^1 x^{-3}\,dx$

Euler's constant γ is defined as

$$\gamma = \lim_{n \to \infty} 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n.$$

It is not clear a priori that this limit exists, and so Exercises 44 and 45 show how to prove that it does exist. Its value is approximately 0.577215, and a very readable account of research related to γ is given by Julian Havil in his book *Gamma*.

◆ **44.** Show that

$$\ln(n+1) \leqslant 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leqslant \ln n,$$

and conclude from this that the sequence $b_n = 1 + 1/2 + 1/3 + \cdots + 1/n - \ln n$ is bounded both above and below.

◆ **45.** Show that

$$\ln(n+1) - \ln n > \frac{1}{n+1},$$

and use this to conclude that the sequence $\{b_n\}$ from Exercise 44 is decreasing. This will imply that $\{b_n\}$ is a decreasing bounded sequence, so its limit, γ, exists by the Monotone Sequence Theorem.

Exercises 46 and 47 show one way to sum the alternating harmonic series, using Euler's constant γ discussed in Exercises 44 and 45.

◆ **46.** Let s_n denote the nth partial sum of the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. Verify that

$$\begin{aligned} s_{2n} &= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n}\right) \\ &\quad - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right). \end{aligned}$$

◆ **47.** Use Exercise 46 to show that

$$s_{2n} - \ln 2 = s_{2n} - (\ln 2n + \gamma) + (\ln n + \gamma) \to 0,$$

proving that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2.$

Exercises 48–50 ask you to develop estimates for $n!$. A more precise estimate is named for James Stirling (1692–1770).

Stirling's Formula. $\displaystyle\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\,(n/e)^n} = 1.$

◆ **48.** Using the fact that $\ln n! = \ln 2 + \ln 3 + \cdots + \ln n$, prove that

$$\ln(n-1)! < \int_1^n \ln x\,dx < \ln n!$$

Note that $\displaystyle\int \ln x\,dx = x\ln x - x + C.$

◆ **49.** Prove that $n! > e\,(n/e)^n$.

◆ **50.** Prove that $n! < en\,(n/e)^n$.

Exercises 51–55 consider a back-of-the-envelope calculation of the escape velocity from Earth using improper integration. These exercises are due to Professor Stephen Greenfield.

51. The continental US is about 3400 miles wide (at its widest point) and contains 4 time zones. Since there are 24 time zones in the world, show that the radius of the Earth is about 4000 miles.

52. Two masses attract each other with a force whose magnitude is proportional to the product of the masses divided by the square of the distance between them. So for a mass m, the magnitude of the force of gravity is GmM/r^2, where G is a constant, M is the mass of the Earth, and r is the distance to the center of the Earth. Since work is equal to force times distance, show that the amount of work needed to lift a mass m from the surface of the Earth to altitude R is

$$\int_{4000}^{R} \frac{GmM}{r^2}\,dr,$$

and calculate this integral. Then let $R \to \infty$ to show that $GmM/4000$ is the most work you can do to lift the mass m to anywhere in the universe (disregarding all objects besides Earth, of course).

53. Using the fact that kinetic energy is $mv^2/2$, compute how much kinetic energy we would need to supply to lift the mass m to anywhere in the universe.

54. Use the fact that acceleration due to gravity on the surface of the Earth is about 32 ft/sec^2, which is equal to $GmM/4000^2$ to solve for GM.

55. Use the answers to Exercises 53 and 54 to show that the escape velocity from the Earth is about 7 miles per second.

ANSWERS TO SELECTED EXERCISES, SECTION 2.4

1. $\displaystyle\int_2^\infty \frac{dx}{x^2} < \sum_{n=1}^\infty \frac{1}{n^2} < 1 + \int_1^\infty \frac{dx}{x^2}$
3. $\displaystyle\int_1^{11} \frac{dx}{\sqrt{x^7+2}} < \sum_{n=1}^{10} \frac{1}{\sqrt{n^7+2}} < \int_0^{10} \frac{dx}{\sqrt{x^7+2}}$
5. Converges by the Integral Test.
7. The Integral Test does not apply (some terms are negative).
9. Converges by the Integral Test (make a u-substitution).
11. The Integral Test does not apply (the series is not decreasing). However, this series diverges by the Test for Divergence.
13. Diverges by the Integral Test (make a u-substitution).
15. The Integral Test does not apply (the series is not decreasing). However, this series diverges by the Test for Divergence.
17. Diverges by the Integral Test (set $u = \ln x$ to evaluate the integral).
19. Diverges by the Integral Test (set $u = \ln \ln x$ to evaluate the integral).
21. 4 terms suffice.
23. e^{100} terms are enough (note that this is 2.688×10^{43})
25. The total area inside the circles is finite.