2. Infinite Series

2.1. A Pre-Requisite: Sequences

We concluded the last section by asking what we would get if we considered the "Taylor polynomial of degree ∞ for the function e^x centered at 0",

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

As we said at the time, we have a lot of groundwork to consider first, such as the fundamental question of what it even means to add an infinite list of numbers together. As we will see in the next section, this is a delicate question. In order to put our explorations on solid ground, we begin by studying sequences.

A *sequence* is just an ordered list of objects. Our sequences are (almost) always lists of real numbers, so another definition for us would be that a sequence is a real-valued function whose domain is the positive integers. The sequence whose nth term is a_n is denoted $\{a_n\}$, or if there might be confusion otherwise, $\{a_n\}_{n=1}^{\infty}$, which indicates that the sequence starts when $n = 1$ and continues forever.

Sequences are specified in several different ways. Perhaps the simplest way is to specify the first few terms, for example

$$\{a_n\} = \{2, 4, 6, 8, 10, 12, 14, \dots\},$$

is a perfectly clear definition of the sequence of positive even integers. This method is slightly less clear when

$$\{a_n\} = \{2, 3, 5, 7, 11, 13, 17, \dots\},$$

although with a bit of imagination, one can deduce that a_n is the nth prime number (for technical reasons, 1 is *not* considered to be a prime number). Of course, this method completely breaks down when the sequence has no discernible pattern, such as

$$\{a_n\} = \{0, 4, 3, 2, 11, 29, 54, 59, 35, 41, 46, \dots\}.$$

To repeat: the above is *not* a good definition of a series. Indeed, this sequence is the number of home runs that Babe Ruth hit from 1914 onward; if the Red Sox had been able to predict the pattern, would they have sold his contract to the Yankees after the 1919 season?

Another method of specifying a sequence is by giving a formula for the *general (nth) term*. For example,

$$\{a_n\} = \{2n\}$$

is another definition of the positive even integers, while $\{a_n\} = \{n^2\}$ defines the sequence $\{a_n\} = \{1, 4, 9, 16, 25, \dots\}$ of squares.

Example 1. Guess a formula for the general term of the sequence

$$\left\{\frac{3}{1}, -\frac{5}{4}, \frac{7}{9}, -\frac{9}{16}, \frac{11}{25}, \dots\right\}.$$

Solution. It is good to tackle this problem one piece at a time. First, notice that the sequence alternates in sign. Since the sequence begins with a positive term, this shows that we should have a factor of $(-1)^{n-1}$ in the formula for a_n. Next, the numerators of these fractions list all the odd numbers starting with 3, so we guess $2n + 1$ for the numerators. The denominators seem to be the squares, so we guess n^2 for these. Putting this together we have $a_n = (-1)^{n-1}(2n + 1)/n^2$. ●

A sequence grows *geometrically* (or *exponentially*) if each term is obtained by multiplying the previous term by a fixed *common ratio*, typically denoted by r. Therefore, letting a denote the first term of a geometric sequence, the sequence can be defined as $\{ar^n\}_{n=0}^{\infty}$; note here that the subscripted $n = 0$ indicates that the sequence starts with $n = 0$. Geometric sequences are also called "geometric progressions".

There is a legendary (but probably fictitious) myth involving a geometric sequence. It is said that when the inventor of chess (an ancient Indian mathematician, in most accounts) showed his invention to his ruler, the ruler was so pleased that he gave the inventor the right to name his prize. The inventor asked for 1 grain of wheat for the first square of the board, 2 grains for the second square, 4 grains for the third square, 8 grains for the fourth square, and so on. The ruler, although initially offended that the inventor asked for so little, agreed to the offer. Days later, the ruler asked his treasurer why it was taking so long to pay the inventor, and the treasurer pointed out that to pay for the 64th square alone, it would take

$$2^{64} = 18,446,744,073,709,551,616$$

grains of wheat. To put this in perspective, a single grain of wheat contains about $1/5$ of a calorie, so 2^{64} grains of wheat contain approximately $3,719,465,121,917,178,880$ calories. Assuming a 2000 calorie per day diet, the amount of wheat *just for the* 64*th square of the chessboard* would feed 6 billion people for almost 850 years† (although they should probably supplement their diet with Vitamin C to prevent scurvy). The legend concludes with

†*Wolfram Alpha* (`http://www.wolframalpha.com/`) is an incredible resource for performing calculations such as this; simply type "2^64 grains of wheat" into its search box and wonder at the results.

the ruler insisting that the inventor participate in the grain counting, in order to make sure that it is "accurate," an offer which the inventor presumably refused.

While sequences are interesting in their own right, we are mostly interested in applying tools for sequences to our study of infinite sums. Therefore, the two most important questions about sequences for us are:

Does the sequence converge or diverge?
If the sequence converges, what does it converge to?

Intuitively, the notion of convergence is often quite clear. For example, the sequence

$$\{0.3, 0.33, 0.333, 0.3333, \dots\}$$

converges to $1/3$, while the sequence

$$\{1/n\} = \{1/1, 1/2, 1/3, 1/4, \dots,\}$$

converges to 0. Slightly more formally, the sequence $\{a_n\}$ converges to the number L if by taking n large enough, we can make the terms of the sequence as close to L as we like. By being a bit more specific in this description, we arrive at the formal definition of convergence below.

Converges to L. The sequence $\{a_n\}$ is said to *converge to* L if for every number $\epsilon > 0$, there is some number N so that $|a_n - L| < \epsilon$ for all $n \geqslant N$.

To indicate that the sequence $\{a_n\}$ converges to L, then we may write

$$\lim_{n\to\infty} a_n = L,$$

or simply

$$a_n \to L \text{ as } n \to \infty.$$

Before moving on, we check one of our previous observations using the formal definition of convergence.

Example 2. Show, using the definition, that the sequence $\{1/n\}$ converges to 0.

Solution. Let $\epsilon > 0$ be any positive number. We want to show that

$$|a_n - 0| = 1/n < \epsilon$$

for sufficiently large n. Solving the above inequality for n, we see that $|a_n - 0| < \epsilon$ for all $n > 1/\epsilon$, proving that $1/n \to 0$ as $n \to \infty$. ●

In practice, we rarely use the formal definition of convergence for examples such as this. After all, the numerator of $1/n$ is constant and the denominator increases without bound, so it is clear that the limit is 0. Many types of limits can be computed with this reasoning and a few techniques, as we show in the next two examples.

Example 3. Compute $\lim_{n\to\infty} \frac{7n+3}{5n+\sqrt{n}}$.

Solution. A common technique with limits is to divide both the numerator and denominator by the fastest growing function of n involved in the expression. In this case the fastest growing function of n involved is n (choosing $7n$ would work just as well), so we divide by n:

$$\lim_{n\to\infty} \frac{7n+3}{5n+\sqrt{n}} = \lim_{n\to\infty} \frac{n}{n} \cdot \frac{7+3/n}{5+\sqrt{n}/n} = \lim_{n\to\infty} \frac{7+3/n}{5+1/\sqrt{n}}.$$

As $n \to \infty$, $3/n \to 0$ and $1/\sqrt{n} \to 0$, so the limit of this sequence is $7/5$. ●

Example 4. Determine the limit as $n \to \infty$ of the sequence $\{\sqrt{n+2} - \sqrt{n}\}$.

Solution. Here we use another frequent technique: when dealing with square roots, it is often helpful to multiply by the "conjugate":

$$\sqrt{n+2} - \sqrt{n} = \left(\sqrt{n+2} - \sqrt{n}\right) \left(\frac{\sqrt{n+2} + \sqrt{n}}{\sqrt{n+2} + \sqrt{n}}\right) = \frac{2}{\sqrt{n+2} + \sqrt{n}}.$$

Now we can analyze this fraction instead. The numerator is constant and the denominator grows without bound, so the sequence converges to 0. ●

You should convince yourself that a given sequence can converge to at most one number. If the sequence $\{a_n\}$ does not converge to any number, then we say that it *diverges*. There are two different types of divergence, and it is important to distinguish them. First we have the type of divergence exhibited by the sequence $\{n!\}$:

> **Diverges to Infinity.** The sequence $\{a_n\}$ is said to *diverge to infinity* if for every number $\ell > 0$, there is some number N so that $a_n > \ell$ for all $n \geqslant N$.

If the sequence $\{a_n\}$ diverges to infinity, then we write $\lim_{n\to\infty} a_n = \infty$. Note that the sequence $\{(-1)^n\} = \{-1, 1, -1, 1, \dots\}$ demonstrates a different type of divergence, which is sometimes referred to as "oscillatory divergence." Both types of divergence show up in our next example.

Example 5. Showing, using only the definition of convergence, that:

(a) $\lim_{n\to\infty} r^n = \infty$ if $r > 1$,

(b) $\lim_{n\to\infty} r^n = 1$ if $r = 1$,

(b) $\lim_{n\to\infty} r^n = 0$ if $-1 < r < 1$ (i.e., if $|r| < 1$),

(c) $\lim_{n\to\infty} r^n$ does not exist if $r \leqslant -1$.

Solution. Beginning with (a), assume that $r > 1$ and fix a number $M > 1$. By taking logarithms, we see that $r^n > M$ if and only if $\ln r^n > \ln M$, or equivalently, if and only if

$$n \ln r > \ln M.$$

Since both M and r are greater than 1, $\ln r, \ln M > 0$. This shows that $r^n > \ln M$ for all $n > \ln M / \ln r$, proving that $\ln r^n = \infty$ when $r > 1$.

Part (b) is obvious, as $1^n = 1$ for all n.

For part (c), let us assume that $|r| < 1$. We want to prove that $\lim_{n\to\infty} r^n = 0$, which means (according to our definition above) that for every $\epsilon > 0$, there is some number N so that

$$|r^n - 0| = |r^n| = |r|^n < \epsilon$$

for all $n \geqslant N$. Again taking logarithms, this holds if $n \ln |r| < \ln \epsilon$. Since $|r| < 1$, its logarithm is negative, so when we divide both sides of this inequality by $\ln |r|$ we flip the inequality. This $|r|^n < \epsilon$ for all $n > \ln \epsilon / \ln |r|$, proving that $\lim_{n\to\infty} r^n = 0$ when $|r| < 1$.

This leaves us with (d). We have already observed that the sequence $\{(-1)^n\}$ diverges, so suppose that $r < -1$. In this case the sequence $\{r^n\}$ can be viewed as a "shuffle" of two sequences, the negative sequence $\{r, r^3, r^5, \dots\}$ and the positive sequence $\{r^2, r^4, r^6, \dots\}$. By part (a), $r^{2n} = (r^2)^n \to \infty$ as $n \to \infty$, while $r^{2n+1} = r \cdot r^{2n} \to -\infty$ as $n \to \infty$, so in this case the sequence $\{r^n\}$ does not have a limit. ●

Another way to specify a sequence is with initial conditions and a recurrence. For example, the factorials† can be specified by the *recurrence*

$$a_n = n \cdot a_{n-1} \text{ for } n \geqslant 2,$$

and the *initial condition* $a_1 = 1$.

†If the inventor of chess had been extremely greedy, he would have asked for 1! grains of wheat for the first square, 2! grains of wheat for the second square, 3! grains of wheat for the third square, and so on, because

$$\begin{aligned} 64! \quad = \quad & 126,886,932,185,884,164,103,433,389,335,161,480,802,865,516, \\ & 174,545,192,198,801,894,375,214,704,230,400,000,000,000,000, \end{aligned}$$

almost 127 octovigintillion, and about a billion times the estimated number of atoms in the universe.

A more complicated recurrence relation is provided by the Fibonacci numbers $\{f_n\}_{n=0}^{\infty}$, defined by

$$f_n = f_{n-1} + f_{n-2} \text{ for } n \geqslant 2,$$

and the initial conditions $f_0 = f_1 = 1$. This sequence begins

$$\{f_n\}_{n=0}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots\}.$$

We now make an important definition. In this definition, note that we twist the notion of "increasing" a bit; what we call increasing sequences should really be called "nondecreasing sequences", but this awkward term is rarely used.

Monotone Sequences. The sequence $\{a_n\}$ is *(weakly) increasing* if $a_n \leqslant a_{n+1}$ for all n, and *(weakly) decreasing* if $a_n \geqslant a_{n+1}$ for all n. A sequence is *monotone* if it is either increasing or decreasing.

It is good to practice identifying monotone sequences. Here are a few examples to practice on:

- $\{1/n\}$ is decreasing,
- $\{1 - 1/n\}$ is increasing,
- $\left\{\dfrac{n}{n^2+1}\right\}$ is decreasing.

You should also verify that the geometric sequence $\{ar^n\}$ is decreasing for $0 < r < 1$, increasing for $r > 1$, and not monotone if r is negative.

It is frequently helpful to know that a given sequence converges, even if we do not know its limit. We have a powerful tool to establish this for monotone sequences. First, we need another definition.

Bounded Sequences. The sequence $\{a_n\}$ is *bounded* if there are numbers ℓ and u such that $\ell \leqslant a_n \leqslant u$ for all n.

For monotone sequences, boundedness implies convergence:

The Monotone Convergence Theorem. Every bounded monotone sequence converges.

We delay the proof of this theorem to Exercise 92. Knowing that a limit exists can sometimes be enough to solve for its true value, as our next example demonstrates.

Example 6. Prove that the sequence defined recursively by $a_1 = 2$ and

$$a_{n+1} = \frac{2a_n}{1 + a_n}$$

for $n \geqslant 1$ is decreasing, and use the Monotone Convergence Theorem to compute its limit.

Solution. First we ask: for which values of n is $a_n \geqslant a_{n+1}$? Substituting the definition of a_{n+1}, this inequality becomes

$$a_n \geqslant \frac{2a_n}{1 + a_n},$$

which simplifies to $a_n^2 \geqslant a_n$. So we have our answer: $a_n \geqslant a_{n+1}$ whenever $a_n \geqslant 1$. We are given that $a_1 = 2$, so $a_1 \geqslant a_2$. For the other values of n, notice that if $a_n \geqslant 1$, then

$$a_{n+1} = \frac{2a_n}{1 + a_n} = \frac{a_n + a_n}{1 + a_n} \geqslant \frac{1 + a_n}{1 + a_n},$$

so since $a_1 \geqslant 1$, we see that $a_2 \geqslant 1$, which implies that $a_3 \geqslant 1$, and so on. In the end, we may conclude that $a_n \geqslant 1$ for *all* values of n. By our work above, this shows that $a_n \geqslant a_{n+1}$ for all values of n, so the sequence is decreasing.

When dealing with monotone sequences, a common technique is to first prove that the sequence has a limit, and then use this fact to find the limit. In order to prove that $\{a_n\}$ has a limit, we need to show that it is bounded. From our previous work, we know that $a_n \geqslant 1$ for all n, and an upper bound is also easy:

$$a_{n+1} = \frac{2a_n}{1 + a_n} \leqslant \frac{2a_n}{a_n} = 2.$$

Therefore, $1 \leqslant a_n \leqslant 2$ for all n, so the sequence $\{a_n\}$ has a limit.

Let L denote this limit. Then we have

$$a_{n+1} = \frac{2a_n}{1 + a_n} \to L$$

as $n \to \infty$. But as $n \to \infty$, $a_n \to \infty$ as well, so we must have

$$\frac{2L}{1 + L} = L,$$

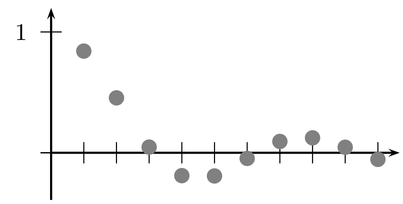
which simplifies to $L^2 = L$. There are two possible solutions, $L = 0$ and $L = 1$, but we can rule out $L = 0$ because it lies outside of our bounds, so it must be the case that this sequence converges to 1. ●

For the rest of the section, we study some other methods for computing limits. One technique to find the limit of the sequence $\{a_n\}$ is to "sandwich" it between a lower bound $\{\ell_n\}$ and an upper bound $\{u_n\}$.

The Sandwich Theorem. Suppose the sequences $\{a_n\}$, $\{\ell_n\}$, and $\{u_n\}$ satisfy $\ell_n \leqslant a_n \leqslant u_n$ for all large n and $\ell_n \to L$ and $u_n \to L$ then $a_n \to L$ as well.

The Sandwich Theorem[†] hopefully seems intuitively obvious. We ask the reader to give a formal proof in Exercise 94. Examples 7 and 8 illustrate its use.

Example 7. Compute $\lim\limits_{n\to\infty} \dfrac{\sin n}{n}$.

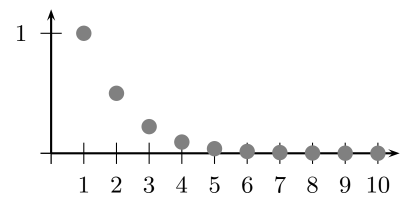


Solution. For all n we have

$$-\frac{1}{n} \leqslant \frac{\sin n}{n} \leqslant \frac{1}{n},$$

so since $-1/n \to 0$ and $1/n \to 0$, $\dfrac{\sin n}{n} \to 0$ by the Sandwich Theorem. ●

Example 8. Show that $\lim\limits_{n\to\infty} \dfrac{n!}{n^n} = 0$.



Solution. Evaluating this limit merely requires finding the right bound:

$$\frac{n!}{n^n} = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \cdots \cdot \frac{2}{n} \cdot \frac{1}{n}, \leqslant \frac{1}{n}$$

so $n!/n^n$ is sandwiched between 0 and $1/n$. Since $1/n \to 0$ and 0 *is* 0, we see by the Sandwich Theorem that $n!/n^n \to 0$. ●

The following useful result allows us to switch from limits of sequences to limits of functions.

Theorem. If $\{a_n\}$ is a function satisfying $a_n = f(n)$ and $\lim\limits_{x\to\infty} f(x)$ exists, then $\lim\limits_{n\to\infty} a_n = \lim\limits_{x\to\infty} f(n)$.

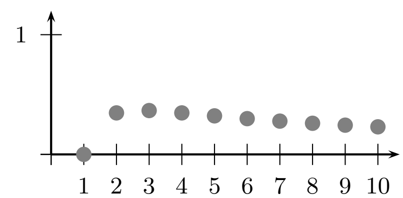
Proof. If $\lim\limits_{n\to\infty} f(x) = L$ then for each $\epsilon > 0$ there is a number N such that $|f(x) - L| < \epsilon$ whenever $x > N$. Of course, this means that $|a_n - L| < \epsilon$ whenever $n > N$, proving the theorem. □

[†] In 1942, Arther Stone and John Tukey proved a theorem called the Ham Sandwich Theorem, which states that given any sandwich composed of bread, ham, and cheese, there is some plane (i.e., straight cut) that slices the sandwich into two pieces, each containing the same amount of bread, the same amount of ham, and the same amount of cheese.

Why might we want to switch from sequences to functions? Generally we do so in order to invoke l'Hôpital's Rule[†]:

> **l'Hôpital's Rule.** Suppose that c is a real number or $c = \pm\infty$, and that f and g are differentiable and $g'(x) \neq 0$ near c (except possibly at $x = c$). If either
>
> $$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$$
>
> or
>
> $$\lim_{x\to c} f(x) = \pm \lim_{x\to c} g(x) = \pm\infty$$
>
> then
>
> $$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)},$$
>
> provided that this limit exists.

Example 9. Compute $\lim_{n\to\infty} \dfrac{\ln n}{n}$.



Solution. We know that

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x},$$

and we can use l'Hôpital's Rule to evaluate this second limit since both $\ln x$ and x tend to ∞ as $x \to \infty$:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

This shows that $\lim_{n\to\infty} \dfrac{\ln n}{n} = 0$. ●

Another very useful result allows us to "move" limits inside continuous functions:

> **The Continuous Function Limits Theorem.** Suppose the sequence $\{a_n\}$ converges to L and that f is continuous at L and defined for all values a_n. Then the sequence $\{f(a_n)\}$ converges to $f(L)$.

[†]l'Hôpital's Rule is one of the many misnomers in mathematics. It is named after Guillaume de l'Hôpital (1661–1704) because it appeared in a calculus book he authored (the first calculus book ever written, in fact), but l'Hôpital's Rule was actually discovered by his mathematics tutor, Johann Bernoulli (1667–1748). The two had a contract entitling l'Hôpital to use Bernoulli's discoveries however he wished.

We conclude the section with two examples using this theorem.

Example 10. Compute $\lim_{n\to\infty} \sin\left(\frac{\pi}{2} - \frac{1}{n}\right)$.

Solution. Since $1/n \to 0$ as $n \to \infty$,

$$\lim_{n\to\infty} \frac{\pi}{2} - \frac{1}{n} = \frac{\pi}{2}.$$

Therefore,

$$\lim_{n\to\infty} \sin\left(\frac{\pi}{2} - \frac{1}{n}\right) = \sin\left(\frac{\pi}{2}\right) = 1,$$

by the Continuous Function Limits Theorem. ●

Our final example is less straight-forward, but more important.

Example 11. Compute $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$.

Solution. As with most problems which have a variable in the exponent, it is a good idea to rewrite the limit using e and $\ln$:

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n\to\infty} e^{\ln\left(1+\frac{1}{n}\right)^n} = \lim_{n\to\infty} e^{n\ln\left(1+\frac{1}{n}\right)}$$

Consider only the exponent for now, $n\ln\left(1 + \frac{1}{n}\right)$. This is an $\infty \cdot 0$ indeterminate form, so we rewrite it to give a $\frac{0}{0}$ form:

$$n\ln\left(1 + \frac{1}{n}\right) = \frac{\ln\left(1 + \frac{1}{n}\right)}{\frac{1}{n}}.$$

Applying l'Hôpital's Rule, we see that

$$\lim_{n\to\infty} \frac{\ln\left(1 + \frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{-\frac{1}{n^2}}{-\frac{1}{n^2}} = 1.$$

Thus the exponents converge to 1. Because e^x is a continuous function, we can now apply the Continuous Function Limits Theorem to see that

$$\lim_{n\to\infty} e^{n\ln\left(1 + \frac{1}{n}\right)} = e^{\lim_{n\to\infty} n\ln\left(1 + \frac{1}{n}\right)} = e^1 = e,$$

so the solution to the example is e. ●

A wonderful resource for integer sequences is the *On-Line Encyclopedia of Integer Sequences*, maintained by Neil Sloane (1939–), available at

`http://www.research.att.com/~njas/sequences/`

The encyclopedia currently contains more than 165,000 sequences. Each of these sequences is numbered, and so the sequence also contains a sequence $\{a_n\}$ (number 91,967) in which a_n is the nth term of the nth sequence. What is a_{91967}?

EXERCISES FOR SECTION 2.1

Write down a formula for the general term, a_n, of each of the sequences in Exercises 1–8.

1. $3, 5, 7, 9, 11, \ldots$

2. $4, 16, 64, 256, 1024, \ldots$

3. $2, -4, 8, -16, 32, \ldots$

4. $5/2, 7/5, 9/10, 11/17, 13/26, \ldots$

5. $3, 1, 3, 1, 3, \ldots$

6. $-4/3, 7/9, -10/27, 13/81, -16/243, \ldots$

7. $1/2, 2/4, 6/8, 24/16, 120/32, \ldots$

8. $2, 4 \cdot 2, 6 \cdot 4 \cdot 2, 8 \cdot 6 \cdot 4 \cdot 2, 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2, \ldots$ (Try to find something simpler than $2n \cdot (2n-2) \cdots 4 \cdot 2$).

Exercises 9–12 give initial terms and recurrence relations for sequences. Use these to compute the first 5 terms and try to write a formula for the general term, a_n.

9. $a_1 = 2$, $a_n = a_{n-1} + 3$

10. $a_1 = 2$, $a_n = n a_{n-1}$

11. $a_1 = 1$, $a_n = a_{n-1} + n$

12. $a_1 = 1$, $a_n = n^3 a_{n-1}$

13. Suppose that $\{a_n\}$ is a geometric sequence. If $a_2 = 6$ and $a_5 = 162$, what are the possibilities for a_1?

14. Suppose that $\{a_n\}$ is a geometric sequence. If $a_2 = 2$ and $a_4 = 6$, what are the possibilities for a_1?

Determine if the sequences in Exercises 15–22 converge or diverge. If they converge, find their limits.

15. $a_n = \dfrac{7n+5}{4n+2}$

16. $a_n = \dfrac{\sqrt{7n+5}}{\sqrt{4n+2}}$

17. $a_n = \dfrac{7n^3+5}{4n^3+2}$

18. $a_n = \dfrac{7\sqrt[4]{n}+5}{4\sqrt[4]{n}+2}$

19. $a_n = \sqrt{2n+2} - \sqrt{2n}$

20. $a_n = \sqrt{2n+2} + \sqrt{2n}$

21. $a_n = 2\arctan(3n^2)$

22. $a_n = 2\arctan(\sqrt{n}+10)$

Compute the limits in Exercises 23–30.

23. $\displaystyle\lim_{n\to\infty} \sqrt{\frac{7n+5}{4n+2}}$

24. $\displaystyle\lim_{n\to\infty} \frac{(n+2)!}{n!(3+5n)^2}$

25. $\displaystyle\lim_{n\to\infty} \frac{5n}{\ln(2+3e^n)}$

26. $\displaystyle\lim_{n\to\infty} \frac{(n+2)!}{n^2 n!}$

27. $\displaystyle\lim_{n\to\infty} \frac{\ln n}{\ln 3n}$

28. $\displaystyle\lim_{n\to\infty} e^{1/n}$

29. $\lim_{n\to\infty} \sqrt[n]{2n}$

30. $\lim_{n\to\infty} (7n+3)^{5/n}$

Prove the limit laws states in Exercises 31–34.

31. If $\{a_n\}$ and $\{b_n\}$ are convergent sequences,

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

32. If $\{a_n\}$ and $\{b_n\}$ are convergent sequences,

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n.$$

33. If $\{a_n\}$ and $\{b_n\}$ are convergent sequences,

$$\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right).$$

34. If $\{a_n\}$ is a convergent sequence, then

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$$

for any number c.

Compute the limits in Exercises 35–42. All of these limits are either 0 or infinity, explanation is required, but you need not apply l'Hôpital's Rule.

35. $\lim_{n\to\infty} \frac{n^n}{n!}$

36. $\lim_{n\to\infty} \frac{n!}{4^n}$

37. $\lim_{n\to\infty} \frac{n^{1/4}}{(\ln n)^4}$

38. $\lim_{n\to\infty} \frac{(2n+1)^{10}}{e^n}$

39. $\lim_{n\to\infty} \frac{n}{(n+3)\ln n}$

40. $\lim_{n\to\infty} \frac{n^3+2n}{\sqrt{n^7+2n^6}}$

41. $\lim_{n\to\infty} \frac{n-2}{(\ln n)^{10}}$

42. $\lim_{n\to\infty} \left(\frac{3n^2-n}{4n^2+1}\right)^n$

43. Arrange the functions

$$n, \quad n^n, \quad \ln n, \quad 3^n, \quad n\ln n, \quad 2^{n^2}, \quad \sqrt{n^6+1}$$

in increasing order, so that (for large n) each function is very much larger than the one that it follows.

44. Where does $n!$ fit in the list you made for Exercise 43?

45. Define the sequence $\{a_n\}$ recursively by

$$a_{n+1} = \frac{2a_n}{1+a_n}.$$

Show that if $a_1 = 1/2$ then $\{a_n\}$ is increasing. (Compare this to the result of Example 6).

46. Define the sequence $\{a_n\}$ recursively by

$$a_{n+1} = \frac{3+3a_n}{3+a_n}.$$

Show that if $a_1 = 1$ then $\{a_n\}$ is increasing, while if $a_1 = 2$ then $\{a_n\}$ is decreasing.

In Exercises 47–50, compute the integral to give a simplified formula for a_n and then determine $\lim_{n\to\infty} a_n$.

47. $a_n = \int_1^n \frac{1}{x}\,dx$

48. $a_n = \int_1^n \frac{1}{x^2}\,dx$

49. $a_n = \int_1^n \frac{1}{x\ln x}\,dx$

50. $a_n = \int_1^n \frac{1}{x(\ln x)^2}\,dx$

51. What is $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$? More generally, what is $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$?

52. Let $\{s_n\}$ denote the sequence given by $1/2, 1/2 + 1/4, 1/2 + 1/4 + 1/8, 1/2 + 1/4 + 1/8 + 1/16, \ldots$. Conjecture a formula for s_n. What does this mean for $\lim_{n\to\infty} s_n$?

Recall that the function $f(x)$ is *periodic* with period p if $f(x+p) = f(x)$ for all values of x. Similarly, the sequence $\{a_n\}_{n=1}^{\infty}$ is periodic with period p if $a_{n+p} = a_n$ for all $n \geqslant 1$ and p is the least number with this property. Determine the periods of the sequences in Exercises 53–56.

53. $\{\sin n\pi\}_{n=1}^{\infty}$

54. $\left\{\cos \dfrac{n\pi}{m}\right\}_{n=1}^{\infty}$

55. $\left\{\cos \dfrac{n\pi}{m} \sin \dfrac{n\pi}{m}\right\}_{n=1}^{\infty}$

56. $\left\{\sin^2 \dfrac{n\pi}{m}\right\}_{n=1}^{m}$

Compute the sequence $f(0), f'(0), f''(0), f^{(3)}(0), \ldots$ of derivatives for the functions $f(x)$ listed in Exercises 57–63.

57. $f(x) = e^x$

58. $f(x) = \sin x$

59. $f(x) = \cos x$

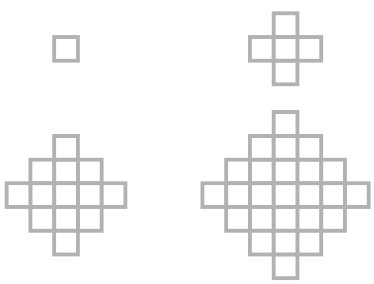
60. $f(x) = \sin 2x$

61. $f(x) = xe^x$

62. $f(x) = (1+x)^3$

63. $f(x) = \sqrt{1+x}$

64. Consider the sequence of figures below.

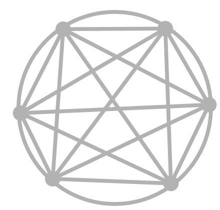


Let a_n denote the number of non-overlapping small squares in these figures, so $a_1 = 1$, $a_2 = 5$, $a_3 = 13$, and $a_4 = 25$. Write a formula for a_n. *Hint:* it may be helpful to consider the squares that are missing from the figures.

Exercises 65 and 66 concern the following sequence. Choose n points on a circle, and join each point to all the others. This divides the circle into a number, say a_n, of regions. For example, $a_1 = 1$ (by definition), $a_2 = 2$, and $a_3 = 4$:

65. Compute a_4 and a_5. Based on the first 5 terms of the sequence, conjecture a formula for the general term a_n.

66. Compute a_6. Does this match your conjecture?



67. (Due to Solomon Golomb (1932–)) There is a unique sequence $\{a_n\}$ of positive integers which is nondecreasing and contains exactly a_n occurrences of the number n for each n. Compute the first 10 terms of this sequence.†

68. Define the sequence $\{a_n\}$ recursively by

$$a_{n+1} = \begin{cases} a_n/2 & \text{if } a_n \text{ is even,} \\ 3a_n + 1 & \text{if } a_n \text{ is odd.} \end{cases}$$

Compute this sequence when $a_1 = 13$ and $a_1 = 24$. The $3n+1$ Conjecture, first posed by Lothar Collatz (1910–1990) in 1937 but still unproved, states that if you start with any positive integer as a_1 then this sequence will eventually reach 1, where it will end in the infinite periodic sequence $1, 4, 2, 1, 4, 2, \ldots$. About this conjecture, the prolific Hungarian mathematician Paul Erdős (1913–1996) said "mathematics is not yet ready for such problems."

69. Compute the first 10 terms of the sequence $a_n = n^2 + n + 41$. What do these numbers have in common?

◆ **70.** Do all terms in the sequence of Exercise 69 share this property?

†Amazingly, a_n is the nearest integer to $\varphi^{2-\varphi} n^{\varphi-1}$ where $\varphi = \dfrac{1+\sqrt{5}}{2}$ denotes the golden ratio.

◆ **71.** It has been known since Euclid (see also Exercises 32–34 in Section 2.3) that there are infinitely many primes‡, but how far apart can they be? Prove that for any positive integer n, the sequence

$$n! + 2, n! + 3, \ldots, n! + n$$

contains *no* prime numbers.

72. Let a_n denote the sum of the integers 1 up to n, so $a_4 = 1 + 2 + 3 + 4 = 10$. Compute the first 6 terms of a_n. Can you give a formula for a_n?

73. Say that a number is *polite* if it can be written as the sum of two or more consecutive positive integers. For example, 14 is polite because $14 = 2 + 3 + 4 + 5$. Let a_n denote the nth polite number. Compute the first 6 terms of a_n. Do you spot a pattern?

74. A *composition* of the integer n is a way of writing n as a sum of positive integers, in which the order of the integers *does* matter. For example, there are eight partitions of 4: 4, $3+1$, $1+3$, $2+2$, $2+1+1$, $1+2+1$, $1+1+2$, and $1+1+1+1$. Let a_n denote the number of compositions of n. Compute the first 6 terms of a_n. Can you conjecture a formula?

75. Let a_n denote the number of compositions of n into 1s and 2s. Relate $\{a_n\}$ to a sequence from this section.

76. For $n \geqslant 2$, let a_n denote the number of compositions of n into parts greater than 1. Relate $\{a_n\}$ to a sequence from this section.

◆ **77.** A *partition* of the integer n is a way of writing n as a sum of positive integers, in which the order of the integers does not matter. For example, there are five partitions of 4: 4, $3+1$, $2+2$, $2+1+1$, and $1+1+1+1$. Let a_n denote the number of partitions of n. Compute the first 6 terms of a_n. Can you conjecture a formula?

◆ **78.** Prove that every infinite sequence $\{a_n\}$ has an infinite monotone subsequence $a_{i_1}, a_{i_2}, a_{i_3}, \ldots$ (with $i_1 < i_2 < \cdots$). *Hint:* a sequence without a greatest element must clearly have an infinite increasing subsequence. So suppose that $\{a_n\}$ has a greatest element, a_{i_1}. Now consider the sequence $a_{i_1+1}, a_{i_1+2}, \ldots$.

◆ **79.** Prove that every sequence $a_1, a_2, \ldots, a_{n^2+1}$ of length $n^2 + 1$ has a monotone subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_{n+1}}$ (with $i_1 < i_2 < \cdots < i_{n+1}$) of length at least $n + 1$. *Hint:* let d_i denote the longest weakly increasing subsequence beginning with a_i, i.e., the largest m so that you can find $a_i = a_{i_1} \leqslant a_{i_2} \leqslant \cdots \leqslant a_{i_m}$ for $i = i_1 < i_2 < \cdots < i_m$. If $d_i \leqslant n$ for all $i = 1, 2, \ldots, n^2 + 1$, how many terms of the sequence must share the same d_i? What does this mean for that subsequence? This result is known as the Erdős-Szekeres Theorem, after Paul Erdős and George Szekeres (1911–2005), who proved it in 1935.

◆ **80.** For each positive integer n, let a_n denote the greatest number which can be expressed as the product of positive integers with sum n. For example, $a_6 = 9$ because $3 \cdot 3$ is the greatest product of positive integers with sum 6. Find a formula for a_n.

Describe the sequences in Exercises 81–84. *Warning:* some of these have very creative definitions. You should use all tools at your disposal, including the Internet.

◆ **81.** $1, 2, 4, 6, 10, 12, 16, 18, \ldots$

◆ **82.** $1896, 1900, 1904, 1906, 1908, 1912, 1920, 1924, \ldots$

◆ **83.** $3, 3, 5, 4, 4, 3, 5, 5, \ldots$

◆ **84.** $1, 2, 3, 2, 1, 2, 3, 4, \ldots$

◆ **85.** How many 0s are there in the decimal expansion of 10!? What about 50! and 100!? *Hint:* the number of 0s is equal to the number of 10s which divide these numbers. These 10s can come by multiplying a number divisible by 10 itself, or by multiplying a number divisible by 5 by a number divisible by 2.

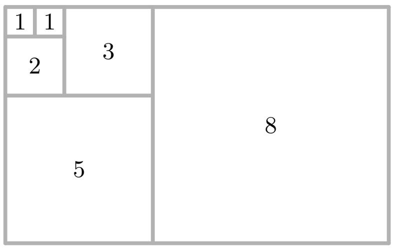
◆ **86.** How many ways are there to order a deck of cards so that each of the suits is together?

‡The distinguished Israeli mathematician Noga Alon recounts:

> "I was interviewed in the Israeli Radio for five minutes and I said that more than 2000 years ago, Euclid proved that there are infinitely many primes. Immediately the host interuppted me and asked: 'Are there still infinitely many primes?"'

◆ **87.** How many ways are there to order a deck of cards so that all of the spades are together? (But the cards of the other suits may be in any order.)

◆ **88.** Use the figure below to conjecture and prove a simplified formula for the sum $\sum_{k=0}^{n} f_k^2$, where f_k denotes the kth Fibonacci number.



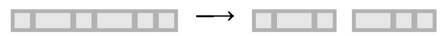
In addition to their recurrence, Fibonacci numbers can also be described in a more concrete way: the nth Fibonacci number f_n counts the number of different ways to *tile* a board of size $n \times 1$ using "squares" of size 1×1 and "dominos" of size 2×1. For example, $f_4 = 5$ because there are four ways to tile a 4×1 board with these pieces:

◆ **89.** Verify that the nth Fibonacci number counts the different ways to tile an $n \times 1$ board using 1×1 squares and 2×1 dominos.

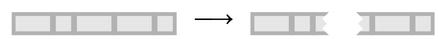
This interpretation can be quite useful in proving identities involving the Fibonacci numbers. Consider, for example, the identity

$$f_{2n} = f_n^2 + f_{n-1}^2.$$

In order to show that this holds for all $n \geqslant 1$, since f_{2n} counts tilings of a $2n \times 1$ board, we need only show that there are $f_n^2 + f_{n-1}^2$ such tilings. Take a particular tiling of a $2n \times 1$ board and chop it into two $n \times 1$ boards. There are two possibilities. First, we might chop the board into two tilings of an $n \times 1$ board, as shown below with an 8×1 board:



Secondly, we might chop through a 2×1 domino, thereby getting two $(n-1) \times 1$ tilings:



The number of $n \times 1$ tilings is f_n, so there are f_n^2 ways in which our chop could break up the $2n \times 1$ tiling in the first manner. Similarly, the number of $(n-1) \times 1$ tilings is f_{n-1}, so there are f_{n-1}^2 in which our chop could break up the $2n \times 1$ tiling in the second manner. Therefore, since we have accounted for all $2n \times 1$ tilings, we see that $f_{2n} = f_n^2 + f_{n-1}^2$, as desired.

Give similar arguments to verify the identities in Exercises 90 and 91.

◆ **90.** Prove that $\displaystyle\sum_{k=0}^{n} f_k = f_{n+2} - 1$.

◆ **91.** Prove that $f_{3n+2} = f_{n+1}^3 + 3f_{n+1}^2 f_n + f_n^3$.

◆ **92.** Prove the Monotone Convergence Theorem in the case where the sequence is monotonically increasing.

◆ **93.** Prove that the converse to the Monotone Convergence Theorem is also true, i.e., that every monotone sequence that is not bounded diverges.

◆ **94.** Prove the Sandwich Theorem.

ANSWERS TO SELECTED EXERCISES, SECTION 2.1

1. $a_n = 2n + 1$
3. $a_n = (-1)^{n-1}2^n$
5. $a_n = \begin{cases} 3 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$
7. $a_n = \dfrac{n!}{2^n}$
9. $\{2, 5, 8, 11, 14, \dots\}$, $a_n = 3n - 1$
11. $\{1, 3, 6, 10, 15, \dots\}$, $a_n = n(n+1)/2$
13. The common ratio must be 3, so $a_1 = 2$.
15. Converges to $7/4$.
17. Converges to $7/4$.
19. Converges to 0.
21. Converges to π.
23. Converges to $\sqrt{7/4}$.