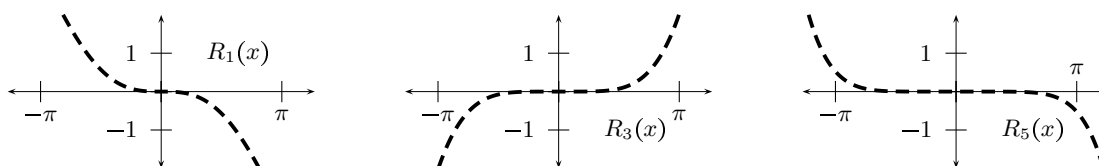


## 1.2. THE REMAINDER THEOREM

Our goal in this section is to explain *why* Taylor polynomials make such nice approximations to certain functions. The object of interest for us is therefore the *remainder term*,

$$R_n(x) = f(x) - T_n(x).$$

While you may be accustomed to “remainder” being a notion from division, here we are using its more general meaning as “the part left over.” Three remainder terms for  $\sin x$  (centered at 0) are shown below. As can be seen from these plots, the Taylor polynomials for  $\sin x$  give better and better approximations near  $x = 0$ .



If we can show that the remainder term is close to 0, then that means that the Taylor polynomial is a good approximation to the function. For example, consider the remainder terms plotted above;  $T_5(x)$  is a better approximation to  $\sin x$  than  $T_3(x)$  because  $R_5(x)$  is close to 0 for more values of  $x$  than  $R_3(x)$ . Our main tool in bounding the remainder term is, somewhat surprisingly, a result from Calculus I.

**Rolle’s Theorem.** Suppose that  $f$  is a differentiable function. If  $f(a) = f(b)$  for some  $a \neq b$  then there is at least one number  $c$  between  $a$  and  $b$  at which  $f'(c) = 0$ .

Rolle’s Theorem is named after the French mathematician Michael Rolle (1652–1719), although it also appears (without proof) in the 12th century work of the Indian astronomer Bhāskara II (circa 1114–1185).

In order to show how Rolle’s Theorem can possibly tell us anything about Taylor polynomials, we begin with the following problem.

**Example 1.** Determine how close the Taylor polynomial of degree 3 for  $\sin x$  centered at 0,  $T_3(x) = x - x^3/6$ , is to  $\sin x$  at the point  $x = 1/2$ .

Of course, there is an easy way to cheat: just compute the difference between  $T_3(1/2)$  and  $\sin 1/2$ . In fact, we did this in the last section. But let’s pretend for a moment that we don’t have access to calculators or computers, so we have to estimate the error, rather than simply compute the error. This is a valuable exercise because error estimates are useful theoretically.

**Solution.** There is certainly some value of  $K$  for which

$$\sin 1/2 = T_3(1/2) + K (1/2)^4,$$

because we could just solve for  $K$  (if we did so, we would find that  $K \approx 0.00414$ , but remember that we are pretending not to have calculators!).

Let  $K$  be this value. Now define the function  $F(x)$  by

$$F(x) = \sin x - T_3(x) - Kx^4.$$

This function is 0 at two points: 0 (which is, not coincidentally, the center), and  $1/2$ , because that's how we defined  $K$ . Therefore we can apply Rolle's Theorem to see that

$$F'(x) = \sin' x - T_3'(x) - 4Kx^3$$

is 0 at some point  $c_1$  between 0 and  $1/2$ . But  $F'(x)$  is also 0 at  $x = 0$ , because  $\sin' x = T_3'(x)$  at  $x = 0$  and  $4Kx^3$  is 0 at  $x = 0$ . Using Rolle's Theorem again, we see that

$$F''(x) = \sin''(x) - T_3''(x) - 12Kx^2$$

is 0 at some point  $c_2$  between 0 and  $c_1$ . Again,  $F''(0) = 0$ , so Rolle's Theorem says that

$$F'''(x) = \sin'''(x) - T_3'''(x) - 24Kx$$

is 0 at some point  $c_3$  between 0 and  $c_2$ . Still,  $F'''(0) = 0$ , so we can apply Rolle's Theorem once again:

$$F^{(4)} = \sin^{(4)}(x) - T_3^{(4)}(x) - 24K$$

is 0 at some point  $c_4$  between 0 and  $c_3$ . This is the point we are really interested in, so let's set  $c = c_4$ . Notice that  $T_3^{(4)}(x)$  is 0, because  $T_3(x)$  is just a degree 3 polynomial, so, solving for  $K$ ,

$$K = \frac{\sin^{(4)}(c)}{24}.$$

But how big can  $K$  be? Since  $\sin^{(4)}(x) = \sin x$ , the numerator of this fraction is between  $-1$  and  $1$ , so  $|K| \leq 1/24$ . So if we go back to how we defined  $K$ , we see that

$$|\sin 1/2 - T_3(1/2)| \leq K (1/2)^4 \leq \frac{1}{24 \cdot 2^4} = \frac{1}{384} \approx 0.0026.$$

(The actual error is a power of 10 smaller, 0.00026.) ●

Thanks to the following theorem, due to Lagrange, we do not have to repeat this calculation over and over again.

**The Remainder Theorem.** Suppose that  $f$  is  $n + 1$  times differentiable and let  $R_n$  denote the difference between  $f(x)$  and the Taylor polynomial of degree  $n$  for  $f(x)$  centered at  $a$ . Then

$$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some  $c$  between  $a$  and  $x$ .

The importance of the Remainder Theorem is that it allows us to bound the size of the remainder easily, based on values of the function's derivatives. Remember that  $n!$  grows very quickly, even more quickly than any exponential, like  $2^n$  or  $10^n$ . This means that the  $(n+1)!$  in the denominator of the Remainder Theorem will (as  $n$  increases) dominate the power  $(x-a)^{n+1}$ . The remainders will therefore get very small, provided that the derivatives don't increase too quickly as  $n$  increases.

The proof of the Remainder Theorem is much like our solution of Example 1.

**Proof of the Remainder Theorem.** We wish to show that for all  $b$ ,

$$R_n(b) = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

for some  $c$  between  $a$  and  $b$ . (We have switched the  $x$ s in the statement into  $b$ s, for clarity in what follows, but of course this is the same result.)

Choose  $K$  so that

$$f(b) = T_n(b) + K(b-a)^{n+1}$$

and define the function

$$F(x) = f(x) - T_n(x) - K(x-a)^{n+1}.$$

Note that the first  $n$  derivatives of  $F(x)$  at  $x = a$  are all 0, because  $T_n(x)$  was constructed to match the first  $n$  derivatives of  $f(x)$  at  $x = a$ , and the first  $n$  derivatives of  $K(x-a)^{n+1}$  at  $x = a$  are all 0.

As we just noted,  $F(a) = 0$  (this is the 0th derivative), and we also have (because we chose  $K$  this way)  $F(b) = 0$ . Therefore we can apply Rolle's Theorem to see that  $F'(c_1) = 0$  for some  $c_1$  between  $a$  and  $b$ . Now consider  $F'(x)$ . We have (so long as  $n \geq 1$ ) that  $F'(a) = 0$ , and we have just seen that  $F'(c_1) = 0$ , so applying Rolle's Theorem again, there is some  $c_2$  between  $a$  and  $c_1$  so that  $F''(c_2) = 0$ . We can repeat this argument  $n$  times, finding a  $c_3$  between  $a$  and  $c_2$  so that  $F'''(c_3) = 0$ , and so on, until we find a  $c_n$  between  $a$  and  $c_{n-1}$  so that  $F^{(n)}(c_n) = 0$ .

At this stage we apply Rolle's Theorem a final time, to see that there is some  $c_{n+1}$  between  $a$  and  $c_n$  so that  $F^{(n+1)}(c_{n+1}) = 0$ . But what is  $F^{(n+1)}(x)$ ? It is nothing but

$$F^{(n+1)}(x) = f^{(n+1)}(x) - (n+1)!K$$

(note that the  $n+1$ st derivative of  $T_n(x)$  is 0, because  $T_n(x)$  is a polynomial of degree  $n$ ). Now we solve for  $K$ :

$$K = \frac{f^{(n+1)}(c_{n+1})}{(n+1)!}.$$

Defining  $c = c_{n+1}$  and returning to our definition of  $K$ , we see that

$$f(b) = T_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Subtracting  $T_n(b)$  from both sides completes the proof.  $\square$

Note that in the proof of the Remainder Theorem we have given, we don't care in the end about  $c_1, c_2, \dots, c_n$  at all, it is only  $c_{n+1}$  that matters. This "book-keeping" is common in mathematical arguments; one must often keep track of extra information which is not itself of interest.

Using the Remainder Theorem to bound the error in an approximation is a straightforward procedure:

1. Find an upper-bound,  $M$ , on the absolute value of the  $n+1$ st derivative of  $f$  between  $a$  (the center) and  $x$  (where you want to bound the error).
2. The error  $|R_n(x)|$  is then at most  $\frac{M}{(n+1)!}|x-a|^{n+1}$ .

**Example 2.** What is the maximum possible error if the Taylor polynomial of degree 5 for the function  $f(x) = \sin x$  centered at 0 is used to approximate  $\sin 1/2$ ?

**Solution.** Since

$$f(1/2) = \sin 1/2 = T_5(1/2) + R_5(1/2),$$

we simply want an upper bound on the remainder term,  $R_5(1/2)$ . The Remainder Theorem states that

$$R_5(1/2) = \frac{f^{(6)}(c)}{6!}(1/2)^6,$$

for some  $c$  between 0 and  $1/2$ . Furthermore,  $|f^{(6)}(c)| < 1$  for all  $c$ , because the derivatives of  $\sin$  are simply  $\cos, -\sin, -\cos,$  and  $\sin$ , so

$$|R_5(x)| \leq \frac{1}{6! \cdot 2^6} = \frac{1}{46080} \approx 0.0000217.$$

This compares well with the actual error we found in Example 1 of Section 1.1, which was 0.0000154.  $\bullet$

**Example 3.** What is the maximum possible error if the Taylor polynomial of degree 4 for the function  $f(x) = e^x$  centered at 0 is used to approximate  $e^{0.2}$ ?

**Solution.** As in the previous example, we simply bound the remainder term. We have

$$R_4(0.2) = \frac{f^{(5)}(c)}{5!}(0.2)^5$$

for some  $c$  between 0 and 0.2. Since  $f^{(5)} = e^x$ , the maximum that  $f^{(5)}(c)$  can be for  $c$  between 0 and 0.2 is  $e^{0.2}$ , so the error in this estimate is at most

$$|R_4(0.2)| \leq \frac{e^{0.2}}{5!}(0.2)^5 \approx 0.0000326$$

(the actual error in this estimate is about 0.0000276). One might complain that we used the value of  $e^{0.2}$  in bounding the error. To avoid this, you can instead use the bound  $f^{(5)}(c) < 2$ , which would give a maximum error of about 0.0000533. ●

**Example 4.** What degree Taylor polynomial for  $\sin x$  (centered at 0) must you use in order to approximate  $\sin 1$  to within 0.001 accuracy?

**Solution.** This is a bit different than the previous problems, because we are given the accuracy and must solve for  $n$ . By the Remainder Theorem,

$$|R_n(1)| \leq \left| \frac{\sin^{(n)}(1)}{(n+1)!} \right|.$$

Because all derivatives of  $\sin x$  are bounded between  $-1$  and  $1$ , we have that  $|R_n(1)| \leq 1/(n+1)!$ . Choosing  $n = 5$  might not work, because  $1/6! = 1/720 > 0.001$ , but  $n = 6$  is certainly enough because  $1/7! = 1/5040 < 0.001$ . (Actually,  $T_5(x)$  and  $T_6(x)$  are the same Taylor polynomials in this case.) ●

Taylor polynomials (and the Remainder Theorem) are frequently used to analyze functions. The next two examples show two such uses. We first revisit the Second Derivative Test, and then we prove the more useful  $n$ th Derivative Test.

**The Second Derivative Test.** Suppose  $f(x)$ ,  $f'(x)$ , and  $f''(x)$  are continuous on an open interval containing a critical point  $a$  of  $f(x)$ .

1. If  $f''(a) < 0$  then  $f$  has a local maximum at  $a$ .
2. If  $f''(a) > 0$  then  $f$  has a local minimum at  $a$ .
3. If  $f''(a) = 0$  then the test is inconclusive.

**Proof.** The Taylor polynomial of degree 1 for  $f(x)$  is

$$T_1(x) = f(a) + \frac{f'(a)}{1!}(x-a),$$

and since  $f'(a) = 0$  (because  $a$  is a critical point),  $T_1(x)$  is simply  $f(a)$ . By the Remainder Theorem, we have

$$f(x) = T_1(x) + R_1(x) = f(a) + \frac{f''(c)}{2}(x-a)^2$$

for some  $c$  between  $a$  and  $x$  (note that  $f'(a) = 0$  by the hypotheses). Because  $f''(x)$  is continuous, for  $c$  near  $a$ ,  $f''(c)$  is near  $f''(a)$ . Therefore if  $f''(a) < 0$  then for  $x$  near  $a$ ,  $f''(c)$  will also be negative, so from the Taylor polynomial expansion of  $f(x)$  we see that  $a$  will be a local maximum. The case  $f''(a) > 0$  follows analogously.  $\square$

The Second Derivative Test is not useful if  $f''(a) = 0$ . Consider, for example,  $f(x) = 1 - \cos x^2$  at the point 0. Here we have  $f'(0) = f''(0) = 0$ , so we cannot classify this critical point. In such situations, a higher order tests is required.

**The  $n$ th Derivative Test.** Suppose  $f(x), f'(x), \dots, f^{(n)}(x)$  are continuous on an open interval containing a critical point  $a$  of  $f(x)$ , that  $f'(a) = f''(a) = \dots = f^{(n-1)}(a) = 0$ , and that  $f^{(n)}(a) \neq 0$ .

1. If  $n$  is even and  $f^{(n)}(a) < 0$  then  $f$  has a local maximum at  $a$ .
2. If  $n$  is even and  $f^{(n)}(a) > 0$  then  $f$  has a local minimum at  $a$ .
3. If  $n$  is odd then  $f$  has an inflection point at  $a$ .

**Proof.** As in the proof of the Second Derivative Test, we expand  $f(x)$  as a Taylor Polynomial centered at  $a$ , although in this case we use a degree  $n - 1$  expansion together with its remainder term:

$$f(x) = f(a) + \frac{f^{(n)}(c)}{n!}(x - a)^n.$$

Because  $f^{(n)}$  is continuous (we assumed this in our hypotheses), for  $x$  close enough to  $a$ ,  $f^{(n)}(c)$  will be the same sign as  $f^{(n)}(a)$ . The proof is now identical to the proof of the Second Derivative Test.  $\square$

**Example 5.** Classify the critical point 0 of  $f(x) = 1 - \cos x^2$ .

**Solution.** We see that  $f'(0) = f''(0) = f'''(0) = 0$  but  $f^{(4)}(0) = 12 > 0$ , so the point 0 is a local minimum for this function by the  $n$ th Derivative Test.  $\bullet$

Because Taylor polynomials provide (or at least, seem to provide) better and better approximations as  $n$  increases, what would happen if we simply “set  $n = \infty$ ?” For example, the Taylor polynomial of degree  $n$  for the function  $f(x) = e^x$  centered at 0 is

$$T_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots + \frac{x^n}{n!}.$$

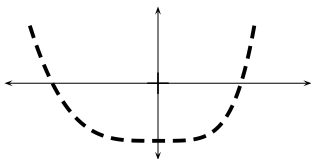
What if we didn't stop at  $n$ ? That would give us the infinite sum

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

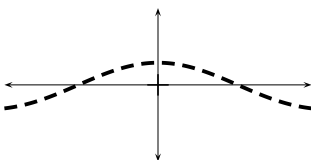
(here  $+\dots$  means to keep adding forever). This is in some sense the “Taylor polynomial of degree  $\infty$  for the function  $f(x) = e^x$  centered at 0.” How well does this infinite sum approximate  $e^x$ ? It turns out, as we will see later, that this infinite sum is actually *equal to*  $e^x$  for all values of  $x$ . Before that though, we need to do quite a bit of background work — what does it even mean to add an infinite list of numbers together?

## EXERCISES FOR SECTION 1.2

**1.** Explain why the graph shown below does not depict the remainder term  $R_n(x)$  for any Taylor polynomial centered at 0.



**2.** Explain why the graph shown below does not depict the remainder term  $R_n(x)$  for any Taylor polynomial centered at 0.



In Exercises 3–6, use the Remainder Theorem to bound the error involved in using the specific Taylor polynomial (centered at 0) to approximate  $f(x)$  at a point.

**3.**  $T_3(x)$  for  $f(x) = \sin x$  at  $x = 0.6$

**4.**  $T_5(x)$  for  $f(x) = \sin x$  at  $x = 0.25$

**5.**  $T_4(x)$  for  $f(x) = e^x$  at  $x = 0.8$

**6.**  $T_3(x)$  for  $f(x) = \cos x$  at  $x = 0.2$

In Exercises 7–14, what degree Taylor polynomial (centered at 0) would be required to approximate  $f(x)$  at the given point to within the stated accuracy? You need not find the best possible degree,

but you should justify your answer using the Remainder Theorem.

**7.**  $f(x) = \sin x$ , at  $x = 0.1$ , to within  $1/100$

**8.**  $f(x) = \sin x$ , at  $x = 0.2$ , to within  $1/100$

**9.**  $f(x) = e^x$ , at  $x = -0.5$ , to within  $1/100$

**10.**  $f(x) = e^x$ , at  $x = -1$ , to within  $1/100$

**11.**  $f(x) = e^{x^2}$ , at  $x = 0.5$ , to within  $1/100$

**12.**  $f(x) = 1 + x^{13}$ , at  $x = -0.1$ , to within  $1/100$

**13.**  $f(x) = (1 + x)^{13}$ , at  $x = 0.1$ , to within  $1/100$

**14.**  $f(x) = x \ln(1 + x)$ , at  $x = -0.2$ , to within  $1/100$

In using Taylor polynomials to approximate a function  $f(x)$  we need to choose the center,  $a$ , to be near  $x$ , but we also need to be able to compute  $f$  and its derivatives at  $a$ . What Taylor polynomial of degree 3 would you use to approximate the functions in Exercises 15–20 at the given value of  $x$ ? Note that these problems have two parts; first you should choose an appropriate center  $a$ , and then you should compute the Taylor polynomials. Also note that there may be several good answers for each problem.

**15.**  $f(x) = \cos x$  near  $x = 0.7$

**16.**  $f(x) = \cos x/7$  near  $x = 22$

**17.**  $f(x) = \sqrt{2+x}$  near  $x = 1.9$

**18.**  $f(x) = \sin x$  near  $x = 2$

**19.**  $f(x) = \sqrt[3]{1+x}$  near  $x = 7.3$

**20.**  $f(x) = \sin x^2$  near  $x = 1.25$

All of the functions in Exercises 21–24 have a critical point at  $x = 0$ . Compute the Taylor polynomial of degree 4 centered at  $a = 0$  and classify this critical point.

21.  $f(x) = \cos x + x^2/2 - 1$

22.  $f(x) = x \sin x^2$

23.  $f(x) = (e^x - 1 - x) \sin x$

24.  $f(x) = x \arctan(x^2)$

A commonly used approximation is

$$1 - x \approx e^{-x}$$

for  $x$  close to 0. Exercises 25–28 investigate this approximation.

25. Approximate  $1 - 1/11$  in this way. Using a calculator or computer, determine the error in your approximation.

26. Approximate  $(1 - 1/11)^5$  in this way. Using a calculator or computer, determine the error in your approximation.

27. Explain the connection between this approximation and Taylor polynomials.

28. For what values of  $x$  is this approximation within  $1/100$  of the true value of  $1 - x$ ?

The result of Exercise 30 is often useful for proving statements about polynomials with only real roots. Exercise 29 is needed for its solution.

◆ 29. If the polynomial  $p(x)$  has a root at  $r$ , i.e.,  $p(r) = 0$ , then the *multiplicity* of  $r$  is the largest integer  $m$  such that  $p(x) = (x - r)^m q(x)$ . Prove that if  $p(x)$  has a root at  $r$  of multiplicity  $m \geq 2$ , then  $p'(x)$  has a root at  $r$  of multiplicity  $m - 1$ .

◆ 30. Using Rolle's Theorem and Exercise 29, prove that if all the roots of the polynomial  $p(x)$  are real, then all of the roots of its derivative are also real.



## ANSWERS TO SELECTED EXERCISES, SECTION 1.2

1. The remainder must be 0 at the center, 0
3. Error  $\leq \frac{0.6^4}{4!}$
5. Error  $\leq \frac{e^{0.8}}{5!}(0.8)^5$
7.  $n = 1$  suffices
9.  $n = 4$  suffices; note that for  $c$  between 0 and  $-0.5$ ,  $e^c$  is at most  $e^0 = 1$ .
11.  $n = 3$  suffices
13.  $n = 5$  suffices
15.  $a = \pi/4 \approx 0.78$  would be a good choice for the center, giving the Taylor polynomial

$$T_3(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{6\sqrt{2}}\left(x - \frac{\pi}{4}\right)^3$$

17.  $a = 2$  would be a good choice for the center, giving the Taylor polynomial

$$T_3(x) = 2 + \frac{1}{4}(x - 2) - \frac{1}{64}(x - 2)^2 + \frac{1}{512}(x - 2)^3$$

19.  $a = 7$  would be a good choice for the center, giving the Taylor polynomial

$$T_3(x) = 2 + \frac{1}{12}(x - 7) - \frac{1}{288}(x - 7)^2 + \frac{5}{20736}(x - 7)^3$$